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Background

Software systems are everywhere. Phones, airplanes, hospitals. Complexity is increasing. Autonomous driving. Manually creating software is very difficult.
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- Complexity is increasing
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- Phones, airplanes, hospitals

Complexity is increasing
- Autonomous driving

Manually creating software is very difficult
Machine Learning to the Rescue

Image recognition, game playing, autonomous driving, etc.

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Machine Learning to the Rescue

Requirements → Input/Output Pairs → Machine Learning Algorithm → Artifact
Machine Learning to the Rescue

- Image recognition, game playing, autonomous driving, etc.
Can Things go Wrong?

Black-box artifacts are useful. Technology is accessible to non-experts. But their opaqueness can be dangerous. Traditional quality-assurance techniques do not apply. Code reviews? Refactoring? Invariants? How do we know what is going on inside the black box?
Can Things go Wrong?

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- How do we know what is going on inside the black box?
When Things go Wrong...
The ACAS Xu System

An Airborne Collision-Avoidance System, for drones

Being developed by the US Federal Aviation Administration

Produce an advisory:

Clear-of-conflict (COC)

Strong left

Weak left

Strong right

Weak right

Ownship $v_{own}$

Intruder $v_{int}$

$\rho$

$\psi$

$\theta$
An *Airborne Collision-Avoidance System*, for drones
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The ACAS Xu System (cnt’d)

ACAS Xu logic too complex for manual implementation

Previous approach: large lookup table (size: 2GB)

Interpolate if needed

Switched to neural networks for compression (size: 3MB)

Also smoother than interpolation

But this requires a new certification procedure

Especially because this is a new approach
ACAS Xu logic *too complex* for manual implementation
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The ACAS Xu System (cnt’d)

Certification via testing and simulation

Encounter plots

But these only cover a finite set of inputs

Verification can help

Guy Katz (HUJI)
Certification via testing and simulation
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*Encounter plots*
The ACAS Xu System (cnt’d)

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Certification via testing and simulation

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Verification

Given program $P$ and property $\varphi$, does $P$ satisfy $\varphi$?

Option 1: prove that property $\varphi$ holds
Option 2: provide a counter-example showing that it does not

Stronger guarantees than testing: holds for any possible input
Not just a finite set that was tested
But, computational cost much higher
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A lot of work on "traditional" systems

Handling common software constructs (e.g., loops, conditions)

Figuring out the properties to check (e.g., no array overflows)

Also, plenty of work on improving scalability

Need to figure this things out for ML-generated software

Is it worth the effort?

Yes, especially for safety-critical systems (like ACAS Xu)
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Adversarial Inputs

In 2014, an intriguing property was observed: Small perturbations of inputs lead to misclassification. Can usually find such inputs very easily.

Guy Katz (HUJI)
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Goodfellow et al., 2015

“panda” 57.7% confidence + $\epsilon \times$ = “gibbon” 99.3% confidence
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Adversarial Inputs

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  \[ \text{“panda” } \quad 57.7\% \text{ confidence} \]
  
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- Small perturbations of inputs lead to misclassification
- Can usually find such inputs very easily

Goodfellow et al., 2015
Adversarial Inputs (cnt’d)

Another example:

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Another example:
Another example:

Traffic Light + 11 White Pixels = Kitchen Oven
Even worse: can cause misclassification to a specific (targeted) input. Attacks can be carried out in the real world. Dangers: Natural malformation of input. Adversary changes “stop” sign into a “entering highway” sign?
Adversarial Inputs (cnt’d)

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Adversarial Inputs (cnt’d)

- Even worse: can cause misclassification to a specific (*targeted*) input
- Attacks can be carried out in the *real world*
- Dangers:
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Adversarial Robustness

A network’s resilience to adversarial attacks is called adversarial robustness. There exist hardening techniques for increasing robustness. But these usually defend against existing attacks and then a new attack breaks them. Verification can be used to establish robustness guarantees.
A network’s resilience to adversarial attacks is called \textit{adversarial robustness}.
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There exist hardening techniques for increasing robustness

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Verification can be used to establish robustness *guarantees*
Roadmap

- Machine-learned software becoming widespread
- Problems with these systems already observed
- Certification is a new and significant challenge

Up next:
- See why neural network verification is hard
- Survey state-of-the-art verification techniques
- Discuss one technique (Reluplex) in more detail
Roadmap

- Machine-learned software becoming widespread
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Verification of ML
UnRAVeL 2019
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Neural Networks

Typical sizes (number of neurons): between few hundreds and millions
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Neural Networks (cnt’d)

First layer is the input layer. In ACAS Xu example: sensor readings.

Final layer is the output layer. In ACAS Xu example: scores for possible advisories.

All other layers are called hidden layers.

Each edge is assigned a weight, and these define the network’s behavior.
First layer is the *input* layer
Neural Networks (cnt’d)

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- Each edge is assigned a *weight*, and these define the network’s behavior
Training Neural Networks

Weights are determined during the training phase: A network is trained on a finite set of inputs... and then expected to generalize to other inputs. Training is about picking good weights: If the network errs, change weights to correct that behavior. Topic of much research, well beyond our scope. We assume that the network has already been trained.
Weights are determined during the *training* phase:
Training Neural Networks

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Evaluating Neural Networks

Nodes evaluated layer by layer:

- Input layer is given
- Every layer computed from its predecessor, according to weights and activation functions

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Verification of ML

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![Diagram of neural network nodes and weights](image.png)
Evaluating Neural Networks

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v_4 = \left( \sum_{i=1}^{3} w_i \cdot v_i \right)
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Evaluating Neural Networks

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Activation Functions

Rectified Linear Unit (ReLU): $f(x) = \max(x, 0)$

Active phase: $x \geq 0$, output is $x$.
Inactive phase: $x < 0$, output is 0.
Activation Functions

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0 \cdot 1 + 2 \cdot 2 + 3 \cdot (-1) = 1
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\[ 1 \cdot 1 + 0 \cdot 2 + 3 \cdot (-1) = -2 \]
Pooling layers:

Max pooling:
$$f(x_1, ..., x_n) = \max(x_1, ..., x_n)$$

Average pooling:
$$f(x_1, ..., x_n) = \frac{1}{n} \sum_{i=1}^{n} x_i$$

Sigmoid function:
$$f(x) = \frac{1}{1 + e^{-x}}$$

Hyperbolic tangent function:
$$f(x) = \tanh(x)$$
Pooling layers:

- Max pooling:
  \[ f(x_1, \ldots, x_n) = \max(x_1, \ldots, x_n) \]

- Average pooling:
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Pool layers:
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- **Sigmoid function:** \( f(x) = \frac{1}{1+e^{-x}} \)

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1. Introduction

2. Neural Networks

3. The Neural Network Verification Problem

4. State-of-the-Art Verification Techniques

5. Reluplex

6. Summary
Neural Network Verification

Definition (The Neural Network Verification Problem)
For a neural network \( \mathcal{N} : \bar{x} \rightarrow \bar{y} \), an input property \( P(\bar{x}) \) and an output property \( Q(\bar{y}) \), does there exist an input \( \bar{x}_0 \) with output \( \bar{y}_0 = \mathcal{N}(\bar{x}_0) \), such that \( \bar{x}_0 \) satisfies \( P \) and \( \bar{y}_0 \) satisfies \( Q \)?

\( P(\bar{x}) \) characterizes the inputs we are checking
\( Q(\bar{y}) \) characterizes undesired behavior for those inputs
Negative answer (\( UNSAT \)) means property holds
Positive answer (\( SAT \)) includes a counterexample
Definition (The Neural Network Verification Problem)

For a neural network $N : \bar{x} \rightarrow \bar{y}$, an input property $P(\bar{x})$ and an output property $Q(\bar{y})$, does there exist an input $\bar{x}_0$ with output $\bar{y}_0 = N(\bar{x}_0)$, such that $\bar{x}_0$ satisfies $P$ and $\bar{y}_0$ satisfies $Q$?
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- $P(\bar{x})$ characterizes the inputs we are checking
- $Q(\bar{y})$ characterizes undesired behavior for those inputs
- Negative answer (UNSAT) means property holds
Neural Network Verification

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For a neural network \( N : \bar{x} \rightarrow \bar{y} \), an input property \( P(\bar{x}) \) and an output property \( Q(\bar{y}) \), does there exist an input \( \bar{x}_0 \) with output \( \bar{y}_0 = N(\bar{x}_0) \), such that \( \bar{x}_0 \) satisfies \( P \) and \( \bar{y}_0 \) satisfies \( Q \)?

- \( P(\bar{x}) \) characterizes the inputs we are checking
- \( Q(\bar{y}) \) characterizes \textit{undesired} behavior for those inputs
- Negative answer (UNSAT) means property \textit{holds}
- Positive answer (SAT) includes a \textit{counterexample}
Example: ACAS Xu

\[\bar{x}[0] \geq 40000\]

\[\bar{y}[0] \leq \bar{y}[1] \lor \bar{y}[0] \leq \bar{y}[2] \lor \bar{y}[0] \leq \bar{y}[3] \lor \bar{y}[0] \leq \bar{y}[4]\]

UNSAT means the system behaves as expected
Example: ACAS Xu

- Want to ensure: whenever intruder is distant, network always answers *clear-of-conflict*
Example: ACAS Xu

- Want to ensure: whenever intruder is distant, network always answers *clear-of-conflict*
- $P(\bar{x})$: 

\[ \bar{x}^0 \geq 40000 \]
\[ \bar{y}^0 \leq \bar{y}^1 \lor \bar{y}^0 \leq \bar{y}^2 \lor \bar{y}^0 \leq \bar{y}^3 \lor \bar{y}^0 \leq \bar{y}^4 \]

**UNSAT** means the system behaves as expected
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- Want to ensure: whenever intruder is distant, network always answers *clear-of-conflict*

- \( P(\bar{x}):\)
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Example: ACAS Xu

- Want to ensure: whenever intruder is distant, network always answers *clear-of-conflict*

  - $P(\bar{x})$: 
    - $\bar{x}[0] \geq 40000$

  - $Q(\bar{y})$: 

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\[ P(\bar{x}): \]
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UNSAT means the system behaves as expected.
Example: ACAS Xu

- Want to ensure: whenever intruder is distant, network always answers *clear-of-conflict*

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Example: Adversarial Robustness

Want to ensure: for a given input $\bar{x}_0$ and a given amount of noise $\delta$, classification remains the same $P(\bar{x})$:

$$\|\bar{x} - \bar{x}_0\|_{L_\infty} \leq \delta$$

Equivalent to:

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Verification Complexity

**Theorem (Neural Network Verification Complexity)**

For a neural network with ReLU activation functions, and for properties $P()$ and $Q()$ that are conjunctions of linear constraints, the verification problem is NP-complete in the number of ReLU nodes.
Verification Complexity

Theorem (Neural Network Verification Complexity)

For a neural network with ReLU activation functions, and for properties $P()$ and $Q()$ that are conjunctions of linear constraints, the verification problem is NP-complete in the number of ReLU nodes.

- Membership in NP: can check in polynomial time that a given $x$ satisfies $P(x)$ and $Q(N(x))$.
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- Membership in NP: can check in polynomial time that a given $x$ satisfies $P(x)$ and $Q(N(x))$
- NP-Hardness: by reduction from 3-SAT
Boolean variables: $x_1, \ldots, x_n$

Input to 3-SAT:

$C_1 \land C_2 \land \ldots \land C_k$

Each clause $C_i$ is $q_1^i \lor q_2^i \lor q_3^i$ where $q_1^i, q_2^i, q_3^i$ are variables or their negations

Goal: find a variable assignment that satisfies the formula

We will construct an input to the verification problem that is satisfiable iff the formula is satisfiable.
Verification Complexity (cnt’d)

- Boolean variables: $x_1, \ldots, x_n$
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Goal: find a variable assignment that satisfies the formula
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Reduction: Handling Negations
Reduction: Handling Negations

- Equation: $q_i^j = \neg x_j$

Diagram:
- Node $x_j$ connected to node $q_i^j$ with a negation arrow.
- Node 1 connected to node $q_i^j$ with a 1 arrow.
- Node 1 connected to node $x_j$ with a 1 arrow.

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Reduction: Handling Negations

$q^j_i$ gets $1 - x_j$, i.e. $q^j_i = \neg x_j$
Reduction: Handling Disjunctions

At least one input is 1: $t_i$ is 0, $y_i$ is 1. All inputs are 0: $t_i$ is 1, $y_i$ is 0. In other words: $y_i = q_1_i \lor q_2_i \lor q_3_i$.
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In other words: $y_i = q_i^1 \lor q_i^2 \lor q_i^3$
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We define the output property, \( Q(y) \), to be

\[ y = n \]

This is satisfied only if all conjuncts are 1.
Reduction: Handling Conjunctions

$y_1 \rightarrow 1$

$\vdots \rightarrow 1$

$y_n \rightarrow y$

$y$ is the final output of the network.

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Reduction: Putting it all Together
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Input property $P(x)$:
$\forall i. x_i \in \{0, 1\}$

Output property $Q(y)$:
$y = n$

Verification property $\text{SAT} \iff$ original formula is SAT

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Verification of ML
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Verification property SAT iff original formula is SAT
Extending the Definition for \( P() \) and \( Q() \)

Corollary

The verification problem remains NP-complete if we allow \( P() \) and \( Q() \) to have arbitrary Boolean structure.

Proof: we add (polynomially many) nodes to handle disjunctions and negations. So, it is enough to solve just for conjunctions.
**Corollary**

*The verification problem remains NP-complete if we allow \( P() \) and \( Q() \) to have arbitrary Boolean structure*
Extending the Definition for P() and Q()

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Another Extension: Max-Pooling

ReLU is a piece-wise linear function
Max-Pooling is also piece-wise linear
Can express one in terms of the other:

$$\text{ReLU}(x) = \max(x, 0)$$

$$\max(x, y) = \text{ReLU}(x - y) + y$$

It is enough to solve just for ReLUs

Other piece-wise linear functions?
Non piece-wise linear functions?
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Neural network verification is hard NP-complete even for simple networks and properties. Real networks can be quite large. So what can we do?

Next, we will:
1. Survey state-of-the-art verification techniques
2. Discuss one such technique (Reluplex) in more detail
Neural network verification is *hard*
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Roadmap

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2. Neural Networks
3. The Neural Network Verification Problem
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6. Summary
**Disclaimer:** The literature on neural network verification is growing rapidly. The work mentioned here is just a sample. Apologies to all authors whose work is not cited.
Techniques and Challenges

Main challenge is scalability

Usually the case in verification

Two kinds of techniques:

- Sound and complete: limited scalability, always succeed
- Sound and incomplete: better scalability, can return “don’t know”

Orthogonal: abstraction techniques

Related: testing techniques (e.g., coverage criteria, concolic testing). Not covered here
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So, How Big a Network can you Verify?

Very difficult to compare!

Different properties make a huge difference

Compare complete and incomplete techniques

Different underlying engines

Different benchmarks

Comparative study: Bunel et al, 2017 [BTT+17]

Still, as a rule of thumb...

Complete techniques: hundreds to thousands

Incomplete techniques: thousands to tens of thousands
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Among first attempts to verify neural networks, NeVeR (Pulina and Tacchella, 2010) focused on networks with Sigmoid activation functions. The main idea was to over-approximate Sigmoids using interval arithmetic and then apply the interval arithmetic solver HySAT.
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Over-Approximations

A common theme in verification is over-approximation. The core idea is to replace a system $S$ with a simpler system $\bar{S}$. All behaviors of $S$ should appear in $\bar{S}$, but additional, spurious behaviors also exist in $\bar{S}$. Because $\bar{S}$ is simpler, it is easier to verify.
Over-Approximations

- A common theme in verification
A common theme in verification

Core idea: replace a system $S$ with a *simpler* $\overline{S}$

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Over-Approximations (cnt’d)

If $\bar{S}$ is correct, so is $S$.

Because all behaviors of $S$ exist in $\bar{S}$.

If $\bar{S}$ is incorrect:

Either $S$ is also incorrect

Or the detected bad behavior is spurious

If needed, $\bar{S}$ is refined to remove the spurious behavior, and the process is repeated.
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NeVeR (Pulina and Tacchella, 2010) [PT10]

For $x \in [a, b]$, we just know that $f(x)$ is in some range $[y_a, y_b]$. When a spurious example is found, the $x$ segments are made smaller, and bounds are made tighter. First step, but could only tackle very small networks (10 neurons).
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  First step, but could only tackle very small networks (10 neurons)
Bastani et al, 2016 [BIL⁺16]

A technique for evaluating a network’s adversarial robustness
A reduction from a verification-like problem to linear programming

Did not directly study verification
But core idea very useful for verification

Guy Katz (HUJI)
A technique for evaluating a network’s adversarial robustness
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- A technique for evaluating a network’s adversarial robustness
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Linear Programming (LP)

A linear program:

\[
\begin{align*}
\text{minimize} & \quad \bar{c} \cdot \bar{x} \\
\text{subject to} & \quad A \cdot \bar{x} = \bar{b} \\
& \quad \bar{l} \leq \bar{x} \leq \bar{u}
\end{align*}
\]

Intuitively:

Set of variables $\bar{x}$, each with lower ($\bar{l}$) and upper ($\bar{u}$) bounds

Set of linear equations that need to hold ($A \cdot \bar{x} = \bar{b}$)

Some objective function to optimize $\bar{c} \cdot \bar{x}$

Highly useful for many problems in CS, studied for many decades

Problem known to be in $P$, powerful solvers exist
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Replacing ReLUs with Linear Constraints

Let $y = \text{ReLU}(x)$. Each ReLU has two phases:

**Active phase:** $(x \geq 0) \land (y = x)$

**Inactive phase:** $(x \leq 0) \land (y = 0)$

Each phase is a linear constraint. True for all piece-wise linear functions, not just ReLUs. If a ReLU is known to be in a specific phase, it can be discarded and replaced with a linear equation.
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If a ReLU is known to be in a specific phase, it can be discarded and *replaced* with a linear equation
To look for adversarial inputs around a point $\bar{x}_0$:

1. Encode the network's weighted sums as linear equations.
2. Evaluate the network for $\bar{x}_0$.
3. For every $y = \text{ReLU}(x)$:
   - If it is active for $\bar{x}_0$, replace it with $(x \geq 0) \land (y = x)$.
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Have an LP solver look for adversarial inputs.

Evaluated on image recognition networks:

- Efficient (LP solvers are fast),
- Sound, but incomplete:
  - Discovered adversarial inputs are correct
  - But may miss some adversarial inputs.
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Reducing Verification to Linear Programming

A complete extension of the technique from Bastani et al

Case splitting: an enumeration of all possibilities:
For each ReLU, guess whether it is active or inactive
Solve the resulting LP
If a solution is found, return SAT
Otherwise, go back and try another guess
If all guesses are exhausted, return UNSAT

Very similar to the naive algorithm for Boolean satisfiability
A complete extension of the technique from Bastani et al
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**Verification of ML**
Reducing Verification to Linear Programming

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Reducing Verification to Linear Programming (cnt’d)
Case splitting creates a *search tree*
Case splitting creates a search tree

Problem is SAT iff at least one leaf is SAT
Case splitting creates a search tree

Problem is SAT iff at least one leaf is SAT

\[ y_1 = \text{ReLU}(x_1), \ y_2 = \text{ReLU}(x_2) \]

\[
\begin{align*}
0 & \quad y_1 = 0, x_1 \leq 0 \\
1 & \quad y_1 = x_1, x_1 \geq 0 \\
\end{align*}
\]

\[
\begin{align*}
1 & \quad y_2 = 0, x_2 \leq 0 \\
2 & \quad y_2 = x_2, x_2 \geq 0 \\
\end{align*}
\]

\[
\begin{align*}
2 & \quad \text{UNSAT} \\
2 & \quad \text{SAT} \\
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Reducing Verification to Linear Programming (cnt’d)

Sound and complete case splitting approach proposed in [KBD+17a].

Approach very sensitive to heuristics and tricks for trimming the search space.

Much like Boolean satisfiability.

Several sound and complete variations, including:

- Ehlers, 2017 [Ehl17] (the Planet solver).
- Tjeng and Tedrake, 2017 [TT17] (the BaB solver).
- Dutta et al, 2018 [DJST18] (the Sherlock solver).
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DLV (Huang et al, 2017) [HKWW17]

Apply a discretization of the input space. Discretization via manipulations. These can represent camera scratches, rotations, etc. Sound but incomplete.

Then do an exhaustive search, layer-by-layer.

Tool: the DLV solver, evaluated on image recognition networks.
Apply a \textit{discretization} of the input space
DLV (Huang et al, 2017) [HKWW17]

- Apply a *discretization* of the input space
  - Discretization via *manipulations*
Apply a *discretization* of the input space

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Tool: the *DLV* solver, evaluated on image recognition networks
AI² (Gehr et al, 2018) [GMDC+18]

Over-approximation of the input property

Over-approximate with polyhedra

Propagate polyhedra layer-by-layer

Sound but incomplete

Abstract property holds $\Rightarrow$ original property holds

Converse not necessarily true
Over-approximation of the *input property*
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Over-approximation of the \textit{input property}
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\textit{Sound} but \textit{incomplete}
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Networks as Continuous Functions

Verification: analyzing this function's properties
Can reduce properties to single output
Analyze a real-valued function
Find lower and upper bounds on the output
The network is a \textit{continuous} function from input to output.
Networks as Continuous Functions

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Verification: analyzing this function’s properties
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  - Analyze a real-valued function

Find lower and upper bounds on the output
DeepGO (Ruan et al, 2018) [RHK18]

Lipschitz Continuity:

\[ |f(x_1) - f(x_2)| \leq K \cdot |x_1 - x_2| \]

*K* is the Lipschitz constant. The best *K* is the smallest one.

Partition input, bound output on each piece, refine if needed.
DeepGO (Ruan et al, 2018) [RHK18]

- **Lipschitz Continuity**: \[ |f(x_1) - f(x_2)| \leq K \cdot |x_1 - x_2| \]

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DeepGO (Ruan et al, 2018) [RHK18] (cnt’d)

Tool: DeepGO

Iteratively refine partition until bounds sufficiently accurate
Guaranteed to converge (complete), assuming a small acceptable error
Smaller values of $K$ lead to faster convergence

Terminate when the discovered bounds imply the property
Complexity also related to size of input domain
Tool: *DeepGO* [RHK18]
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Iteratively refine partition until bounds sufficiently accurate
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Tool: *DeepGO* [RHK18]

Iteratively refine partition until bounds sufficiently accurate
- Guaranteed to converge (*complete*), assuming a small acceptable error
- Smaller values of $K$ lead to *faster* convergence

Terminate when the discovered bounds imply the property
DeepGO (Ruan et al, 2018) [RHK18] (cnt’d)

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Verification of Binarized Neural Networks

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Network reachability analysis via over-approximations around specific inputs
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Neural network verification is hard. NP-complete even for simple networks and properties. Reducible to an exponential sequence of easy problems. Sound and complete. Much work on finding efficient heuristics. Can trade completeness for better scalability. Can be combined with abstraction techniques. Next, we will:

1. Focus on one sound and complete technique (Reluplex) in greater detail.
Roadmap

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Reluplex

Joint work with Clark Barrett, David Dill, Kyle Julian and Mykel Kochenderfer (CAV 2017 [KBD+17a]), supported by the FAA and Intel.

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Reluplex (cnt’d)

SMT-solver for quantifier-free linear real arithmetic + ReLUs

Based on the Simplex method for linear programming

Simplex + ReLUs = Reluplex

Applicable to other piece-wise linear functions

Key SMT idea: handle ReLUs lazily

As opposed to eager case splitting

Defer splitting for as long as possible

May not have to split at all!

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Simplex

Developed shortly after WW2 by George Dantzig

An algorithm for solving linear programs

Linear equations
Variable bounds
Objective function

Very efficient, still in use today
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Simplex (cnt’d)

Divided into two phases:
1. Find a feasible solution
2. Optimize with respect to objective function

We focus on phase 1, which is just a satisfiability check.
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Simplex: Phase 1

Iterative algorithm
Always maintain a variable assignment
Assignment always satisfies equations
But may violate bounds
In every iteration, attempt to reduce the overall infeasibility
Simplex: Phase 1

- Iterative algorithm
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- Assignment always *satisfies equations*
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- In every iteration, attempt to reduce the overall *infeasibility*
Variables partitioned into basic and non-basic variables. Non-basics are "free" and basics are "bounded". Non-basic assignment dictates basic assignment. This is how the equations are maintained.

In every iteration, we can perform:
1. An update: change the assignment of a non-basic variable and any affected basics.
2. A pivot: switch a basic and non-basic variable.
Variables partitioned into \textit{basic} and \textit{non-basic} variables
Variables partitioned into *basic* and *non-basic* variables

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Simplex: Example

Hidden layer
Input layer
Output layer

Property being checked: for $x_1 \in [0,1]$, always $x_4 \in [0.5,1]$

Negated output property: $x_1 \in [0,1]$ and $x_4 \in [0.5,1]$
Simplex: Example

Property being checked: for \( x_1 \in [0, 1] \), always \( x_4 / \in [0.5, 1] \).

Negated output property: \( x_1 \in [0, 1] \) and \( x_4 \in [0.5, 1] \).
No activation functions
No activation functions

Property being checked: for $x_1 \in [0, 1]$, always $x_4 \notin [0.5, 1]$
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- Negated output property: $x_1 \in [0, 1]$ and $x_4 \in [0.5, 1]$
Equations for weighted sums:

\[ x_2 - x_1 = 0 \]
\[ x_3 + x_1 = 0 \]
\[ x_4 - x_3 - x_2 = 0 \]

Bounds:

\[ x_1 \in [0, 1] \]
\[ x_4 \in [0.5, 1] \]
\[ x_2, x_3 \text{ unbounded} \]

Technicality: replace constants by auxiliary variables.
Simplex: Example (cnt’d)

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\[ x_2, x_3 \text{ unbounded} \]
\[ x_5, x_6, x_7 \in [0, 0] \]

Technicality: replace constants by *auxiliary* variables
Equations for weighted sums:

\[ x_2 - x_1 = x_5 \]
\[ x_3 + x_1 = x_6 \]
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\[ x_5 = x_2 - x_1 \]
\[ x_6 = x_3 + x_1 \]
\[ x_7 = x_4 - x_3 - x_2 \]

<table>
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<th>Lower B.</th>
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<tr>
<td>0</td>
<td>(x_1)</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>(x_2)</td>
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Simplex: Example (cnt’d)

\[ x_5 = x_2 - x_1 \]
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Update:

\[ x_4 := x_4 + 0.5 \]

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Simplex: Example (cnt’d)

\[x_5 = x_2 - x_1\]
\[x_6 = x_3 + x_1\]
\[x_7 = x_4 - x_3 - x_2\]

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\[ x_5 = x_2 - x_1 \]
\[ x_6 = x_3 + x_1 \]
\[ x_7 = x_4 - x_3 - x_2 \]

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Lower B. & Var & Value & Upper B. \\
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0.5 & \( x_4 \) & 0.5 & 1 \\
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0 & \( x_5 \) & 0 & 0 \\
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\[ x_5 = x_2 - x_1 \]
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Pivot: \( x_7, x_2 \)

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\[ x_5 = x_2 - x_1 \]
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\[ x_2 = x_4 - x_3 - x_7 \]

Pivot: \( x_7, x_2 \)

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\[ x_5 = x_4 - x_3 - x_7 - x_1 \]
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**Update:**
\[ x_7 := x_7 - 0.5 \]

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Update:

\[ x_7 := x_7 - 0.5 \]

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Update:
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\begin{align*}
x_5 &= x_4 - x_3 - x_7 - x_1 \\
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\end{align*}

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\[ x_5 := x_5 - 0.5 \]

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Simplex: Example (cnt’d)

\[ x_1 = x_4 - x_3 - x_7 - x_5 \]
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Update:
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The Simplex Calculus

A simplex configuration:

### Distinguished Symbols

- SAT
- UNSAT

Or a tuple $\langle B, T, l, u, \alpha \rangle$, where:

- $B$: set of basic variables
- $T$: a set of equations
- $l, u$: lower and upper bounds
- $\alpha$: an assignment function from variables to reals

For notation:

- Slack $+ (x_i) = \{x_j / x_j \in B | (T_{i,j} > 0 \land \alpha(x_j) < u(x_j)) \lor (T_{i,j} < 0 \land \alpha(x_j) > l(x_j))\}$
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$$\text{slack}^+(x_i) = \{x_j \notin B \mid (T_{i,j} > 0 \land \alpha(x_j) < u(x_j)) \lor (T_{i,j} < 0 \land \alpha(x_j) > l(x_j))\}$$

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The Simplex Calculus (cnt’d)

Pivot 1

\[ x_i \in B, \alpha(x_i) < l(x_i), x_j \in \text{slack}^+ \]

\[ T := \text{pivot}(T, i, j), B := B \cup \{x_j\} \setminus \{x_i\} \]

Pivot 2

\[ x_i \in B, \alpha(x_i) > u(x_i), x_j \in \text{slack}^- \]

\[ T := \text{pivot}(T, i, j), B := B \cup \{x_j\} \setminus \{x_i\} \]

Update

\[ x_j / \in B, \alpha(x_j) < l(x_j) \lor \alpha(x_j) > u(x_j), l(x_j) \leq \alpha(x_j) + \delta \leq u(x_j) \]

\[ \alpha := \text{update}(\alpha, x_j, \delta) \]

Failure

\[ x_i \in B, (\alpha(x_i) < l(x_i) \land \text{slack}^+ \setminus \{x_i\} = \emptyset) \lor (\alpha(x_i) > u(x_i) \land \text{slack}^- \setminus \{x_i\} = \emptyset) \]

UNSAT

Success

\[ \forall x_i \in X. l(x_i) \leq \alpha(x_i) \leq u(x_i) \]

SAT
The Simplex Calculus (cnt’d)

Pivot

\[ x_i \in B, \quad \alpha(x_i) < l(x_i), \quad x_j \in \text{slack}^+(x_i) \]

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The Simplex Calculus (cnt’d)

\[\begin{align*}
\text{Pivot}_1: & \quad x_i \in B, \quad \alpha(x_i) < l(x_i), \quad x_j \in \text{slack}^+(x_i) \\
& \quad T := \text{pivot}(T, i, j), \quad B := B \cup \{x_j\} \setminus \{x_i\}
\end{align*}\]

\[\begin{align*}
\text{Pivot}_2: & \quad x_i \in B, \quad \alpha(x_i) > u(x_i), \quad x_j \in \text{slack}^−(x_i) \\
& \quad T := \text{pivot}(T, i, j), \quad B := B \cup \{x_j\} \setminus \{x_i\}
\end{align*}\]
The Simplex Calculus (cnt’d)

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Pivot 2  \[ x_i \in B, \quad \alpha(x_i) > u(x_i), \quad x_j \in \text{slack}^-(x_i) \]
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\]

\[
\text{UNSAT}
\]

Success

\[
\forall x_i \in \mathcal{X}. \quad l(x_i) \leq \alpha(x_i) \leq u(x_i)
\]

\[
\text{SAT}
\]
Properties of Simplex

Theorem (Soundness and Completeness of Simplex)
The simplex algorithm is sound and complete*

Soundness:
SAT $\Rightarrow$ assignment is correct
UNSAT $\Rightarrow$ no assignment exists

Completeness: depends on variable selection strategy
Bland’s rule: guarantees termination
Always pick variables with smallest index
Prevents cycling
But unfortunately quite slow
Better selection strategies exist (e.g., steepest edge)

Problem is in $P$, unknown whether simplex is in $P$
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*Guy Katz (HUJI)
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From Simplex to Reluplex

Each ReLU node $x$ represented as two variables:
- $x_w$ to represent the (input) weighted sum
- $x_a$ to represent the (output) activation result

$x_w$ and $x_a$ change independently

May violate ReLU constraints

Similar to bound constraints

Fix incrementally

Use pivots and updates, same as before
From Simplex to Reluplex

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- May violate ReLU constraints
- Similar to bound constraints
- Fix incrementally

Use pivots and updates, same as before
Reluplex: Example

\[
\begin{align*}
\text{ReLU} & \\
&= \max(0, x) \\
&= \begin{cases} \\
0 & \text{if } x \leq 0 \\
x & \text{if } x > 0
\end{cases}
\end{align*}
\]
Reluplex: Example

\[ x_1 \rightarrow x_2 \rightarrow x_4 \]

\[ x_1 \rightarrow x_3 \rightarrow x_4 \]

\[ x_1 \rightarrow \text{ReLU} \rightarrow x_2 \rightarrow \text{ReLU} \rightarrow x_4 \]

\[ x_1 \rightarrow \text{ReLU} \rightarrow x_3 \rightarrow \text{ReLU} \rightarrow x_4 \]
Reluplex: Example (cnt’d)

Equations for weighted sums:

\[ x_5 = x_w^2 - x_1 \]

\[ x_6 = x_w^3 + x_1 \]

\[ x_7 = x_4 - x_a^3 - x_a^2 \]

Bounds:

\[ x_1 \in [0, 1] \]

\[ x_4 \in [0.5, 1] \]

\[ x_w^2, x_w^3 \text{ unbounded} \]

\[ x_a^2, x_a^3 \in [0, \infty) \]

\[ x_5, x_6, x_7 \in [0, 0] \]
Equations for weighted sums:

\[ x_5 = x_w^2 - x_1 \]
\[ x_6 = x_w^3 + x_1 \]
\[ x_7 = x_4 - x_a^3 - x_a^2 \]

Bounds:

\[ x_1 \in [0, 1] \]
\[ x_4 \in [0, 1] \]
\[ x_w^2, x_w^3 \text{ unbounded} \]
\[ x_a^2, x_a^3 \in [0, \infty) \]
\[ x_5, x_6, x_7 \in [0, 0] \]
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\begin{align*}
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Reluplex: Example (cnt’d)

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    x_1 &\in [0, 1] \\
    x_4 &\in [0.5, 1] \\
    x_2^w, x_3^w &\text{ unbounded} \\
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Update:
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Pivot: \( x_7, x_2^a \)

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Pivot: \( x_7, x_2^a \)

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**Update:**
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Reluplex: Example (cnt’d)

\[
x_5 = x^w_2 - x_1 \\
x_6 = x^w_3 + x_1 \\
x^a_2 = x_4 - x^a_3 - x_7
\]

Update:
\[
x_7 := x_7 - 0.5
\]

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\[ x_5 = x_2^w - x_1 \]
\[ x_6 = x_3^w + x_1 \]
\[ x_2^a = x_4 - x_3^a - x_7 \]

\[ \begin{array}{c|c|c|c}
\text{Lower B.} & \text{Var} & \text{Value} & \text{Upper B.} \\
\hline
0 & x_1 & 0 & 1 \\
\hline
& x_2^w & 0 & \\
\hline
0 & x_2^a & 0.5 & \\
\hline
& x_3^w & 0 & \\
\hline
0 & x_3^a & 0 & \\
\hline
0.5 & x_4 & 0.5 & 1 \\
\hline
0 & x_5 & 0 & 0 \\
\hline
0 & x_6 & 0 & 0 \\
\hline
0 & x_7 & 0 & 0 \\
\end{array} \]
Reluplex: Example (cnt’d)

\[
x_5 = x_2^w - x_1 \\
x_6 = x_3^w + x_1 \\
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Reluplex: Example (cnt’d)

\[ x_5 = x_2^w - x_1 \]
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\[ x_2^a = x_4 - x_3^a - x_7 \]

**Update:**
\[ x_2^w := x_2^w + 0.5 \]

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Reluplex: Example (cnt’d)

\[ x_5 = x_2^w - x_1 \]
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Update:
\[ x_2^w := x_2^w + 0.5 \]

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Reluplex: Example (cnt’d)

\[ x_5 = x_2^w - x_1 \]
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Update:
\[ x_2^w := x_2^w + 0.5 \]

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 & x_2^w & 0.5 & \\
\hline
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\hline
 & x_3^w & 0 & \\
\hline
0 & x_3^a & 0 & \\
\hline
0.5 & x_4 & 0.5 & 1 \\
\hline
0 & x_5 & 0.5 & 0 \\
\hline
0 & x_6 & 0 & 0 \\
\hline
0 & x_7 & 0 & 0 \\
\hline
\end{array}
\]
Reluplex: Example (cnt’d)

\[ x_5 = x_2^w - x_1 \]
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Reluplex: Example (cnt’d)

\[ x_5 = x_2^w - x_1 \]
\[ x_6 = x_3^w + x_1 \]
\[ x_2^a = x_4 - x_3^a - x_7 \]

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$$x_5 = x_2^w - x_1$$

$$x_6 = x_3^w + x_1$$

$$x_2^a = x_4 - x_3^a - x_7$$

**Pivot:** $x_5, x_1$

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Reluplex: Example (cnt’d)

\[
x_5 = x_2^w - x_1
\]

\[
x_6 = x_3^w + x_1
\]

\[
x_2^a = x_4 - x_3^a - x_7
\]

Pivot: \(x_5, x_1\)

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Reluplex: Example (cnt’d)

\[ x_1 = x^w_2 - x_5 \]
\[ x_6 = x^w_3 + x^w_2 - x_5 \]
\[ x^a_2 = x_4 - x^a_3 - x_7 \]

Pivot: \( x_5, x_1 \)

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\[ x_1 = x_2^w - x_5 \]
\[ x_6 = x_3^w + x_2^w - x_5 \]
\[ x_2^a = x_4 - x_3^a - x_7 \]

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Reluplex: Example (cnt’d)

\[ x_1 = x_2^w - x_5 \]
\[ x_6 = x_3^w + x_2^w - x_5 \]
\[ x_2^a = x_4 - x_3^a - x_7 \]

Update:
\[ x_5 := x_5 - 0.5 \]

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Reluplex: Example (cnt’d)

\[
\begin{align*}
    x_1 &= x_2^w - x_5 \\
    x_6 &= x_3^w + x_2^w - x_5 \\
    x_2^a &= x_4 - x_3^a - x_7
\end{align*}
\]

Update:

\[x_5 := x_5 - 0.5\]

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Reluplex: Example (cnt’d)

\[ x_1 = x^w_2 - x_5 \]
\[ x_6 = x^w_3 + x^w_2 - x_5 \]
\[ x^a_2 = x_4 - x^a_3 - x_7 \]

Update:
\[ x_5 := x_5 - 0.5 \]

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Reluplex: Example (cnt’d)

\[x_1 = x_2^w - x_5\]
\[x_6 = x_3^w + x_2^w - x_5\]
\[x_2^a = x_4 - x_3^a - x_7\]

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Pivot: \( x_6, x_3^w \)

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**Update:**
\[ x_6 := x_6 - 0.5 \]

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ReLU

Property: \(x_1 \in [0, 1]\) and \(x_4 \in [0.5, 1]\)
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The Reluplex Calculus

A Reluplex configuration:

\[ \langle B, T, l, u, \alpha, R \rangle \]

- \( B \): set of basic variables
- \( T \): set of equations
- \( l, u \): lower and upper bounds
- \( \alpha \): assignment function from variables to reals
- \( R \subset X \times X \): set of ReLU connections
A Reluplex configuration:
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- Distinguished symbols SAT or UNSAT
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- Distinguished symbols SAT or UNSAT
- Or a tuple $\langle B, T, l, u, \alpha, R \rangle$, where:
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Verification of ML
UnRAVeL 2019
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The Reluplex Calculus (cnt’d)

Pivot

1, Pivot 2, Update and Failure are as before

SAT iff at least one leaf of the derivation tree is SAT

Update \( w_x^i / \in B \), \( \langle x_i, x_j \rangle \in R \), \( \alpha(x_j) \neq \max(0, \alpha(x_i)) \), \( \alpha(x_j) \geq 0 \)

\( \alpha := \text{update}(\alpha, x_i, \alpha(x_j) - \alpha(x_i)) \)

Update \( a_x^j / \in B \), \( \langle x_i, x_j \rangle \in R \), \( \alpha(x_j) \neq \max(0, \alpha(x_i)) \)

\( \alpha := \text{update}(\alpha, x_j, \max(0, \alpha(x_i)) - \alpha(x_j)) \)

PivotForRelu

\( x_i \in B \), \( \exists x_l \).

\( \langle x_i, x_l \rangle \in R \lor \langle x_l, x_i \rangle \in R \), \( x_j / \in B \), \( T_{i,j} \neq 0 \)

\( T := \text{pivot}(T, i, j) \), \( B := B \cup \{x_j\} \backslash \{x_i\} \)

ReluSplit

\( \langle x_i, x_j \rangle \in R \), \( l(x_i) < 0 \), \( u(x_i) > 0 \)

\( u(x_i) := 0 \), \( l(x_i) := 0 \)

ReluSuccess

\( \forall x \in X. \ l(x) \leq \alpha(x) \leq u(x) \), \( \forall \langle x_w, x_a \rangle \in R. \ \alpha(x_a) = \max(0, \alpha(x_w)) \)

SAT

Guy Katz (HUJI)
Pivot\textsubscript{1}, Pivot\textsubscript{2}, Update and Failure are as before.
The Reluplex Calculus (cnt’d)

- $\text{Pivot}_1$, $\text{Pivot}_2$, Update and Failure are as before
- SAT iff at least one leaf of the derivation tree is SAT
The Reluplex Calculus (cnt’d)

- Pivot₁, Pivot₂, Update and Failure are as before

- SAT iff at least one leaf of the derivation tree is SAT

\[
\text{Update}_w \quad \begin{array}{c}
  x_i \notin \mathcal{B}, \quad \langle x_i, x_j \rangle \in R, \quad \alpha(x_j) \neq \max (0, \alpha(x_i)), \quad \alpha(x_j) \geq 0 \\
  \alpha := \text{update} (\alpha, x_i, \alpha(x_j) - \alpha(x_i))
\end{array}
\]
Pivot₁, Pivot₂, Update and Failure are as before

SAT iff at least one leaf of the derivation tree is SAT

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The Reluplex Calculus (cnt’d)

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\[
\text{PivotForRelu} \quad \frac{x_i \in B, \ \exists x_l. \ (x_i, x_l) \in R \lor (x_l, x_i) \in R, \ x_j \notin B, \ T_{i,j} \neq 0}{T := \text{pivot}(T, i, j), \ B := B \cup \{x_j\} \setminus \{x_i\}}
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- **Pivot₁, Pivot₂, Update and Failure** are as before

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\text{ReluSuccess} \quad \forall x \in \mathcal{X}. \ l(x) \leq \alpha(x) \leq u(x), \ \forall \langle x^w, x^a \rangle \in R. \ \alpha(x^a) = \max(0, \alpha(x^w)) \quad \text{SAT}
\]
Properties of Reluplex

Theorem (Soundness and Completeness of Reluplex)

The Reluplex algorithm is sound and complete*

Soundness:
SAT $\Rightarrow$ assignment is correct
UNSAT $\Rightarrow$ no assignment exists

Completeness: depends on variable selection strategy and splitting strategy

Naive approach: split on all variables immediately, apply Bland's rule
This is the case-splitting approach from before
Ensures termination
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More Efficient Reluplex

Better approach:
lazy splitting

Start fixing bound violations

Once all variables within bounds, address broken ReLUs

If a ReLU is repeatedly broken, split on it

Otherwise, fix it without splitting

And repeat as needed

Usually end up splitting on a fraction of the ReLUs (20%)

Can reduce splitting further with some additional work
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More Efficient Reluplex: Bound Tightening

During execution we encounter many equations. Can use them for bound tightening. Example:

\[ x = y + z \]
\[ x \geq -2, \quad y \geq 1, \quad z \geq 1 \]

Can derive tighter bound:

\[ x \geq 2 \]

If \( x \) is part of a ReLU pair, we say the ReLUs phase is fixed and we replace it by a linear equation. Same as in case splitting, only no back-tracking required.

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In every pivot step we examine an equation
Use that equation for bound tightening
For the basic variable
For other variables, too?
Complexity: linear in the size of the equation
Particularly useful after splitting
Because new bounds have been introduced
Can be combined with
backjumping
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Non-Chronological Backtracking (Backjumping)

A useful technique in SAT and SMT solving

Backtracking: change last guess

Backjumping: change an earlier guess

Need to keep track of the discovery of new bounds
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Non-Chronological Backtracking (Backjumping) (cnt’d)

\[ y_1 = \text{ReLU}(x_1), \ y_2 = \text{ReLU}(x_2) \]

- \[ y_1 = 0, \ x_1 \leq 0 \]
- \[ y_1 = x_1, \ x_1 \geq 0 \]
- \[ y_2 = 0, \ x_2 \leq 0 \]
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UNSAT
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Precision and Numerical Stability

SMT solvers typically use precise arithmetic. This ensures soundness but is quite slow.

LP solvers typically use floating point arithmetic. Rounding errors can harm soundness but is much faster.

LP solvers attempt to avoid division by tiny fractions. Should do the same when implementing Reluplex.
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Precision and Numerical Stability (cnt’d)

- Can monitor numerical instability
- Plug current assignment into input formulas
- Measure the error
- If the degradation exceeds a certain threshold, restore the equations from the original
- Fewer pivot operations, and hence more accuracy
- Still does not guarantee soundness
- Open question for most techniques
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Roadmap

- The simplex algorithm, for solving linear programs
- Extension into Reluplex, for solving linear programs + ReLUs

Some highlights for an efficient implementation

Up next:
- We will talk about use-cases where Reluplex was applied
  1. ACAS Xu Verification
  2. Adversarial Robustness
  3. Reluplex + Clustering
Roadmap

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The ACAS Xu System

An Airborne Collision-Avoidance System, for drones
Being developed by the US Federal Aviation Administration (FAA)

Produce an advisory:

- Clear-of-conflict (COC)
- Strong left
- Weak left
- Strong right
- Weak right

Ownship

Intruder

$\rho$

$\psi$

$\theta$

Implemented using neural networks

Guy Katz (HUJI)

Verification of ML

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There are properties that the FAA cares about:

1. Consistent alerting regions
2. No unnecessary turning advisories
3. Strong alerts do not occur when intruder vertically distant

These properties are defined formally.

Constraints on inputs and outputs.
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- Properties defined formally
  - Constraints on inputs and outputs
We worked on a list of 10 properties

Example 1:
If the intruder is near and approaching from the left, the network advises strong right

Distance:
\[ 12000 \leq \rho \leq 62000 \]

Angle to intruder:
\[ 0.2 \leq \theta \leq 0.4 \]

Etc.

Proved in less than 1.5 hours, using 4 machines
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Example 2:
If vertical separation is large and the previous advisory is weak left, the network advises clear-of-conflict or weak left.

Distance: \[0 \leq \rho \leq 60\]

Time to loss of vertical separation: \[\tau = 100\]

Etc.

Found a counter-example in 11 hours.

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- If vertical separation is large and the previous advisory is weak left, the network advises clear-of-conflict or weak left
  - Distance: $0 \leq \rho \leq 60760$
  - Time to loss of vertical separation: $\tau = 100$
  - Etc.
- Found a counter-example in 11 hours
Certifying ACAS Xu (cnt’d)
<table>
<thead>
<tr>
<th></th>
<th>Networks</th>
<th>Result</th>
<th>Time</th>
<th>Stack</th>
<th>Splits</th>
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<td>463</td>
<td>55</td>
<td>88388</td>
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<td>35</td>
<td>SAT</td>
<td>82419</td>
<td>44</td>
<td>284515</td>
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<tr>
<td>$\phi_3$</td>
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<td>28156</td>
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<td>52080</td>
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<tr>
<td>$\phi_4$</td>
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<td>$\phi_7$</td>
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<td>$\phi_8$</td>
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<td>19944</td>
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<td>88520</td>
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</tbody>
</table>
Adversarial Robustness

Slight perturbations of inputs lead to misclassification. Verification can prove that this cannot occur, allowing us to assess attacks and defenses.

Guy Katz (HUJI)
Adversarial Robustness

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Goodfellow et al., 2015

“panda”
57.7% confidence

+ $\epsilon \times$

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- Slight perturbations of inputs lead to misclassification
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- Allows us to assess attacks defenses

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Local Adversarial Robustness

Verification question: for a given panda $\bar{x}$ and a given amount of noise $\delta$, does classification remain the same?

If $\|\bar{x} - \bar{x}_0\|_{L} \leq \delta$ then $\bigwedge_i (\bar{y}_i[0] \geq \bar{y}_i)$, where $\bar{y}_i[0]$ is the desired label.

Easiest norm to handle: $L_\infty$, the infinity norm

$\|\bar{x} - \bar{x}_0\|_{L_\infty} \leq \delta \iff \forall i. -\delta \leq \bar{x}_i[i] - \bar{x}_0[i] \leq \delta$

Can also handle $L_1$: $\|\bar{x} - \bar{x}_0\|_{L_1} \leq \delta \iff \sum_{n=1}^{\infty} |\bar{x}_i[i] - \bar{x}_0[i]| \leq \delta$

And we know that $\max(a, b) = \text{ReLU}(a - b) + b$
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Can also handle $L_1$: $\|\bar{x} - \bar{x}_0\|_{L_1} \leq \delta \iff \sum_{n=1}^{\mathbf{i}} |\bar{x}_i[0] - \bar{x}_0[i]| \leq \delta |\bar{x}_i[0] - \bar{x}_0[i]| = \max(\bar{x}_i[0] - \bar{x}_0[i], \bar{x}_0[i] - \bar{x}_0[i])$.
Verification question: for a given panda $\bar{x}_0$ and a given amount of noise $\delta$, does classification remain the same?

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Easiest norm to handle: \( L_\infty \), the infinity norm.
Local Adversarial Robustness

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- And we know that $\max(a, b) = \text{ReLU}(a - b) + b$
Local Adversarial Robustness (cnt’d)

Can find the optimal $\delta$ for which robustness holds

Using binary search

Example: an ACAS Xu network

<table>
<thead>
<tr>
<th>$\delta$</th>
<th>Result</th>
<th>Time</th>
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</thead>
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<tr>
<td>0.025</td>
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<td>0.05</td>
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<td>24</td>
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<td>1</td>
<td>SAT</td>
<td>135</td>
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</table>

Guy Katz (HUJI)
Verification of ML
UnRAVeL 2019 103 / 116
Can find the \textit{optimal} $\delta$ for which robustness holds
Local Adversarial Robustness (cnt’d)

- Can find the *optimal* $\delta$ for which robustness holds
  - Using binary search
Can find the *optimal* $\delta$ for which robustness holds
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Example: an ACAS Xu network
Local Adversarial Robustness (cnt’d)

- Can find the \textit{optimal} $\delta$ for which robustness holds
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- Example: an ACAS Xu network

<table>
<thead>
<tr>
<th></th>
<th>$\delta = 0.1$</th>
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<th>$\delta = 0.05$</th>
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<td>Result Time</td>
<td>Result Time</td>
<td>Result Time</td>
<td>Result Time</td>
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<td>SAT 24</td>
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</table>
Assessing Attacks and Defenses [CKBD18]

Assessing attacks:

Pick point $\bar{x}$

Use verification to find optimal $\delta$

Use attack to find $\delta'$

See how close $\delta'$ is to $\delta$

Example: Carlini-Wagner attack [CW17] on a small MNIST network

On average, $\delta$ is about 6% smaller than $\delta'$
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On average, $\delta$ about 6% smaller than $\delta'$
Assessing defenses:
Start with network N
Train hardened network \( \bar{N} \)
Pick point \( \bar{x} \)
Compare optimal \( \delta \) before and after hardening

Example: Madry defense [MMS+18] on a small MNIST network
On average, hardened \( \delta \) about 423% larger
However, smaller in some cases
Assessing defenses:

Start with network $N$

Train hardened network $\overline{N}$

Pick point $\overline{x}$

Compare optimal $\delta$ before and after hardening

Example: Madry defense [MMS +18] on a small MNIST network

On average, hardened $\delta$ is about $423\%$ larger

However, smaller in some cases
Assessing defenses:
- Start with network $N$

Example: Madry defense [MMS+18] on a small MNIST network

On average, hardened $\delta$ about $42\%$ larger
However, smaller in some cases
Assessing defenses:
- Start with network $N$
- Train *hardened* network $\bar{N}$
Assessing defenses:

- Start with network $N$
- Train \textit{hardened} network $\tilde{N}$
- Pick point $\bar{x}$
Assessing defenses:

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- Train hardened network $\tilde{N}$
- Pick point $\tilde{x}$
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Global Robustness?

Previous definition: for a particular input $\bar{x}_0$.

What's an acceptable $\delta$? How do you pick $\bar{x}_0$?

Can you evaluate the overall robustness?
Global Robustness?

- Previous definition: for a particular input $\bar{x}_0$
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Let $p_1, p_2$ be confidence levels for certain label:

$$\forall \bar{x}_1, \bar{x}_2. \|\bar{x}_1 - \bar{x}_2\| \leq \delta \Rightarrow |p_1 - p_2| \leq \epsilon$$

Small changes to input do not change output by much

Significantly slower to compute

Double the network size

Large input regions

And also still need to choose $\delta, \epsilon$

A compromise: a clustering based approach
Global Robustness Queries

- Region boundaries: look at *confidence* instead of label
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- A compromise: a *clustering* based approach
DeepSafe: A Clustering-Based Approach [GKPB18]

Use clustering to identify regions on which the network should be consistent.

Clustering applied to known points (e.g., training set).

Identify centroid $\bar{x}_0$ and radius $\delta$ for each cluster.

Higher degree of automation.

Discovered an adversarial example in ACAS Xu.
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# Table of Contents

1. Introduction
2. Neural Networks
3. The Neural Network Verification Problem
4. State-of-the-Art Verification Techniques
5. Reluplex
6. Summary
Summary

Software generated by machine learning is becoming widespread. Certifying this software is a new and exciting challenge. Verification can play a key role.

The main questions:

- How do we verify?
- What do we verify?
Software generated by machine learning is becoming widespread.
Software generated by machine learning is becoming *widespread*

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Summary - Approaches to Verification

The sound and complete approaches

An NP-complete problem

Usually based on the case splitting approach

Can be improved with:

- Tighter bound derivation
- Splitting heuristics
- Local optimization steps
The *sound and complete* approaches
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Summary - Approaches to Verification (cnt’d)

Trading completeness for scalability

Discretization and exhaustive search techniques

Correct-by-construction networks

Abstraction techniques

Approximating the network

Approximating the input property
Trading *completeness* for *scalability*
Trading **completeness** for **scalability**

- Discretization and exhaustive search techniques
Trading *completeness* for *scalability*

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- Trading *completeness* for *scalability*
  - Discretization and exhaustive search techniques
  - Correct-by-construction networks

- Abstraction techniques
Summary - Approaches to Verification (cnt’d)

- Trading *completeness* for *scalability*
  - Discretization and exhaustive search techniques
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- Abstraction techniques
  - Approximating the *network*
Trading \textit{completeness} for \textit{scalability}
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Abstraction techniques
- Approximating the \textit{network}
- Approximating the \textit{input property}
Properties to Verify

Domain-specific properties
Example: ACAS Xu
Human input required — a known issue in verification

General properties
Adversarial robustness
Always desirable, regardless of networks
Can we find other such properties?
Properties to Verify

- *Domain-specific* properties
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Ongoing Work in the Reluplex Project

Improving scalability

Currently: linear and non-linear steps roughly independent

Can we solve both kinds of constraints together?

Better SMT techniques?

Proof certificates

Numerical stability is an issue

SAT answers can be checked, but what about UNSAT?

Replay the solution, using precise arithmetic

Generate an externally-checkable proof certificate
Ongoing Work in the Reluplex Project

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Ongoing Work in the Reluplex Project (cnt’d)

More expressiveness
Handle non piece-wise linear activation functions?

Case studies
More extensive verification of ACAS Xu

Systems in which the network is just a component?

Collaboration with various industrial partners

Guy Katz (HUJI)
Ongoing Work in the Reluplex Project (cnt’d)

- More *expressiveness*
More expressiveness

- Handle non piece-wise linear activation functions?
More expressiveness
  - Handle non piece-wise linear activation functions?

Case studies
Ongoing Work in the Reluplex Project (cnt’d)

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Thank You!

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