Automated Run–Time Analysis of Probabilistic Programs

Marcel Hark

Outline

Introduction

Basics of Probabilistic Programs

Semantics of Probabilistic Programs

My Work

Outline

Non–Probabilistic Programs

Example

$x := n$
while $x > 0$ **do**
| $x := x - 1$

Non–Probabilistic Programs

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$\rightarrow$ How to estimate number of loop-iterations?

Non–Probabilistic Programs

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$\rightarrow$ $R : \mathbb{Z}^2 \rightarrow \mathbb{Z}, (x, n) \mapsto x$ is a ranking function for the loop.

Non–Probabilistic Programs

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$x := n$
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$\rightarrow$ R strictly positive

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→ Automated in our tool „KoAT“ [ACM–TOPLAS 2016].

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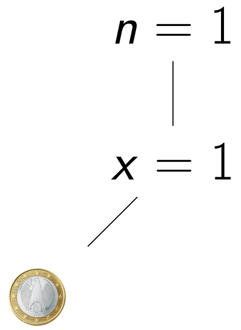

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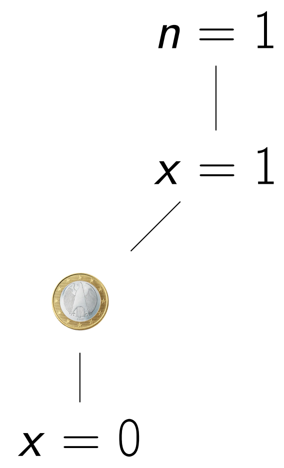

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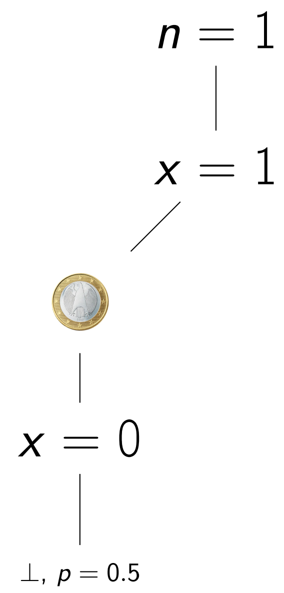

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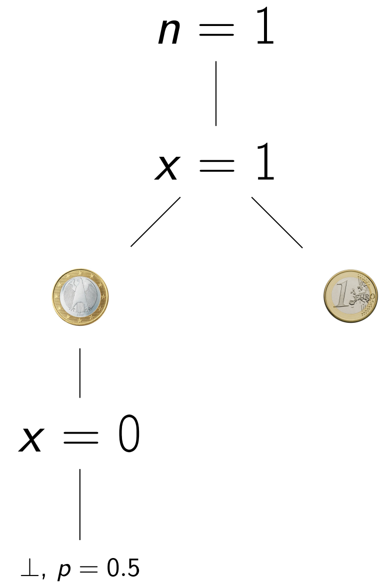

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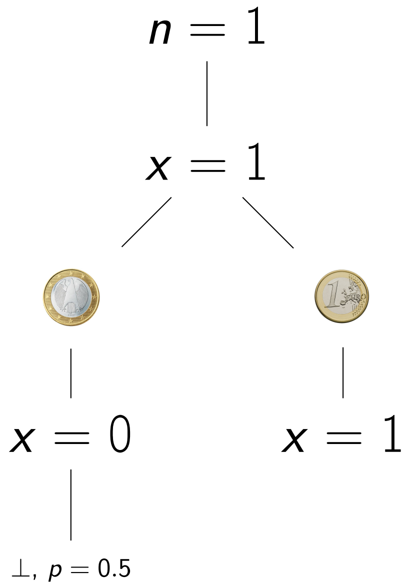

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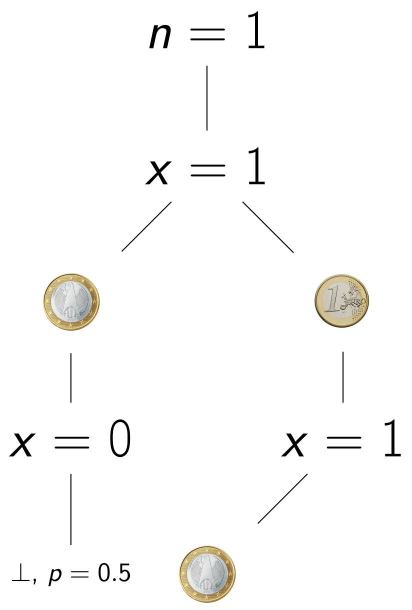

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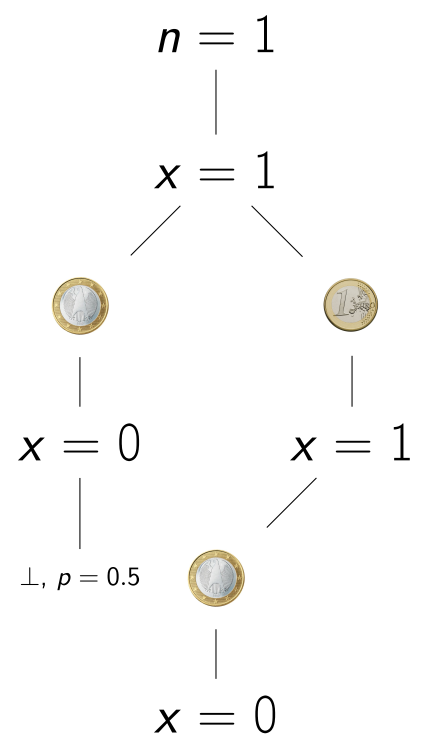

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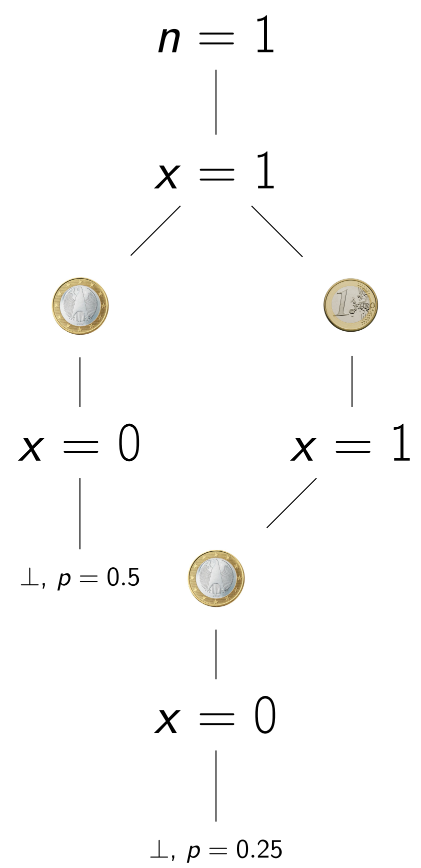

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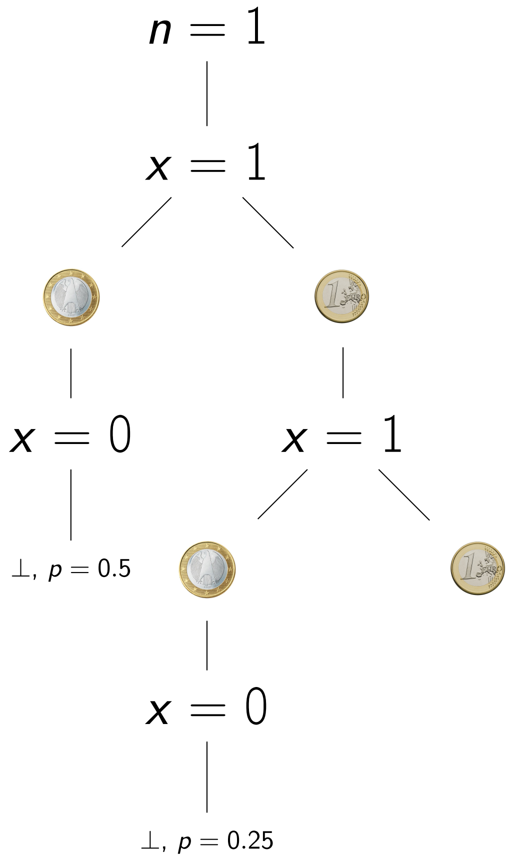

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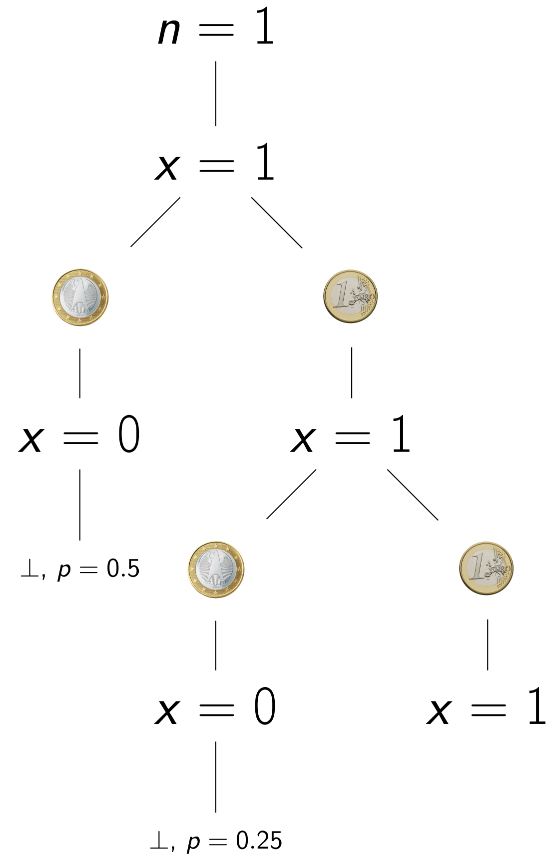

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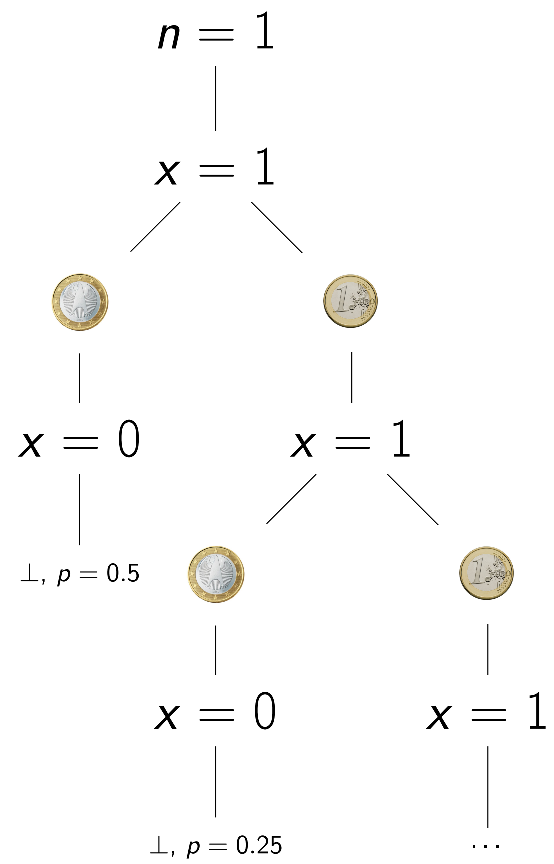

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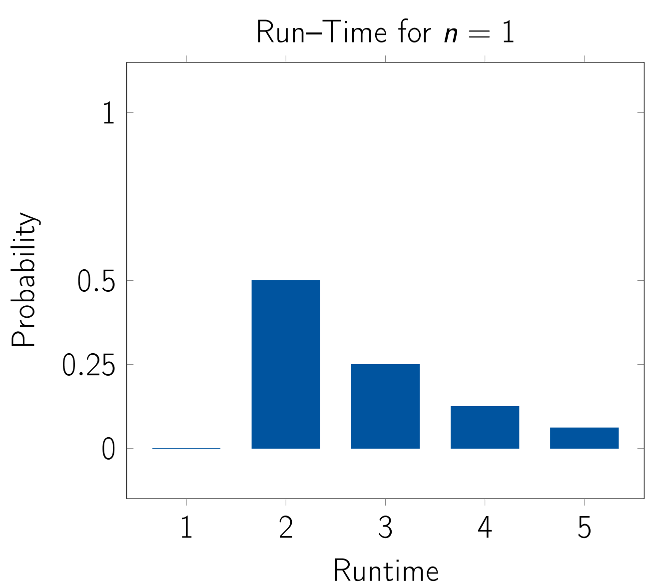

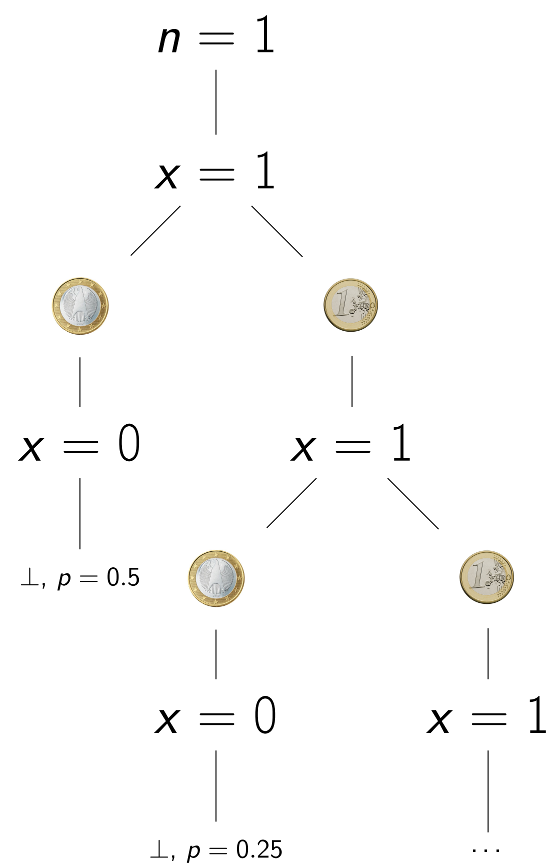

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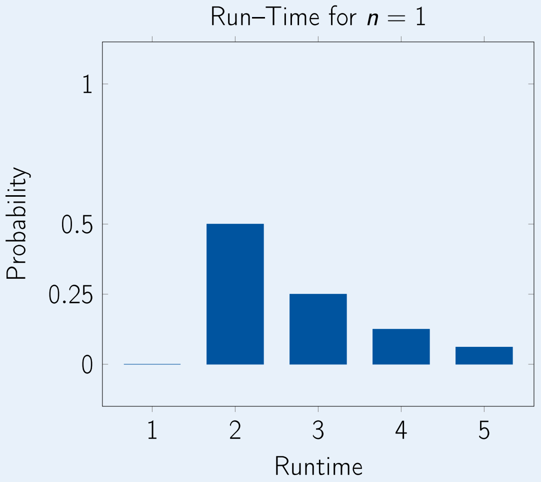

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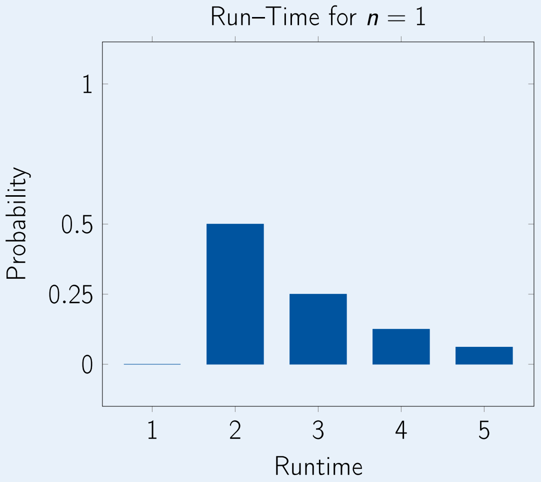

- Runtime is a positive random-variable *Term*

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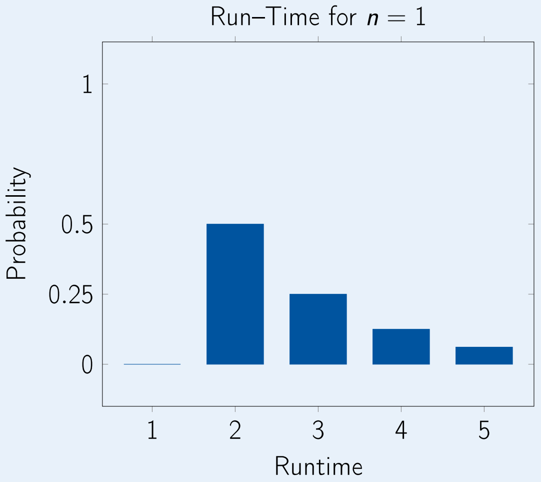

- Runtime is a positive random-variable *Term* $\Rightarrow \mathbb{P}(Term = m + 1) = \left(\frac{1}{2}\right)^m, \; m \geq 1$

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- **positive almost surely** if $\mathbb{E}(\mathit{Term}) < \infty$ (**P.A.S.T.**)

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Theorem

P.A.S.T. implies A.S.T.

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A.S.T. is weaker than P.A.S.T

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Objective

Reason about termination types fully automatically.

Types of Termination

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Objective

Reason about termination types fully automatically.
Compute bounds on $\mathbb{E}(\mathit{Term})$.

Types of Termination

Definition

A probabilistic program terminates

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Remark

A.S.T. and P.A.S.T are **undecidable**.

Outline

Introduction

Basics of Probabilistic Programs

Semantics of Probabilistic Programs

My Work

Probabilistic Control Flow Graph

Example

$x := n$

while $x > 0$ **do**

$\quad x := \frac{1}{2}\langle x - 1\rangle + \frac{1}{2}\langle x\rangle$

Probabilistic Control Flow Graph

Example

$x := n$
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Semantics of Probabilistic Programs

Probabilistic Control Flow Graph

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Probabilistic Control Flow Graph

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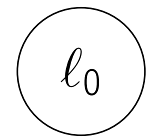

Probabilistic Control Flow Graph

Example

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ℓ_1

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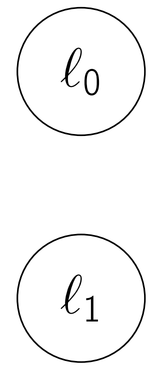

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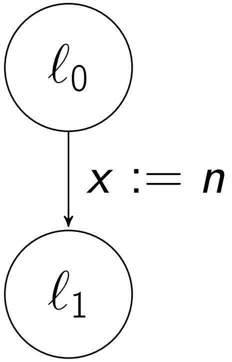

Probabilistic Control Flow Graph

Example

(ℓ_0)

$x := n$

(ℓ_1)

while $x > 0$ **do**

  (ℓ_2)

  if $\mathit{prob}(0.5)$ **then**

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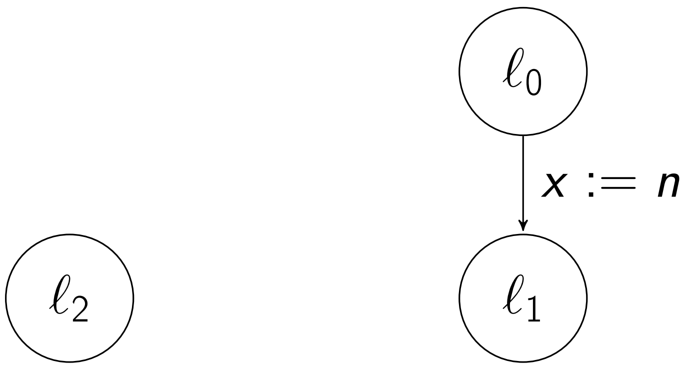
    $x := x$



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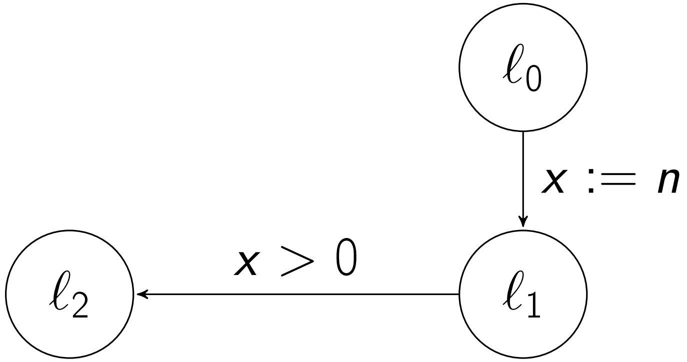

Probabilistic Control Flow Graph

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  ℓ_2

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  else

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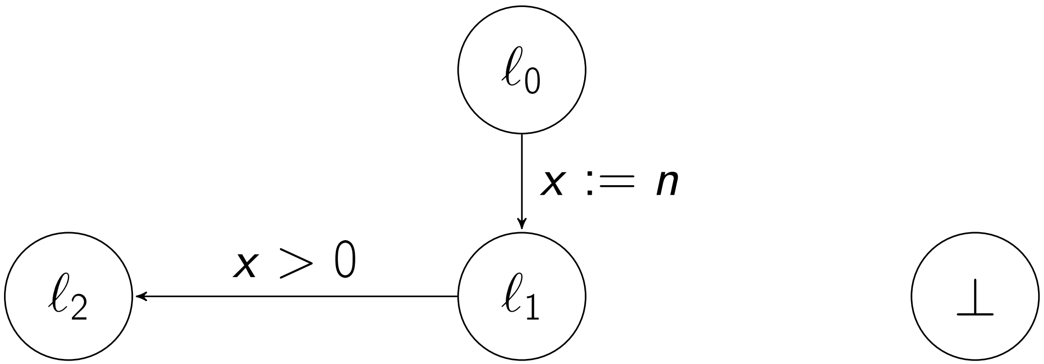
$\bot$



Probabilistic Control Flow Graph

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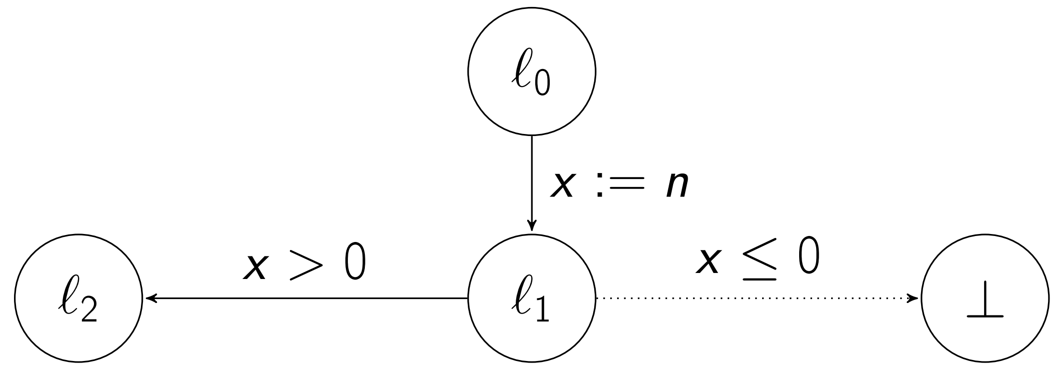

Probabilistic Control Flow Graph

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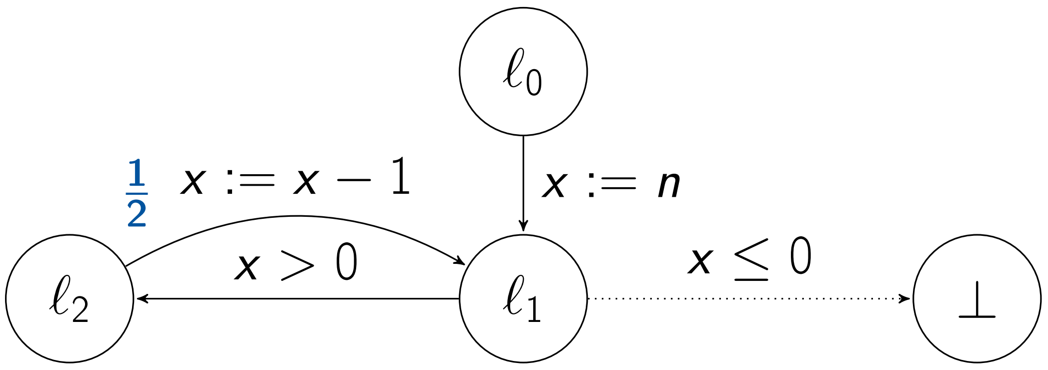


Probabilistic Control Flow Graph

Example

ℓ_0

$x := n$

ℓ_1

while $x > 0$ **do**

  ℓ_2

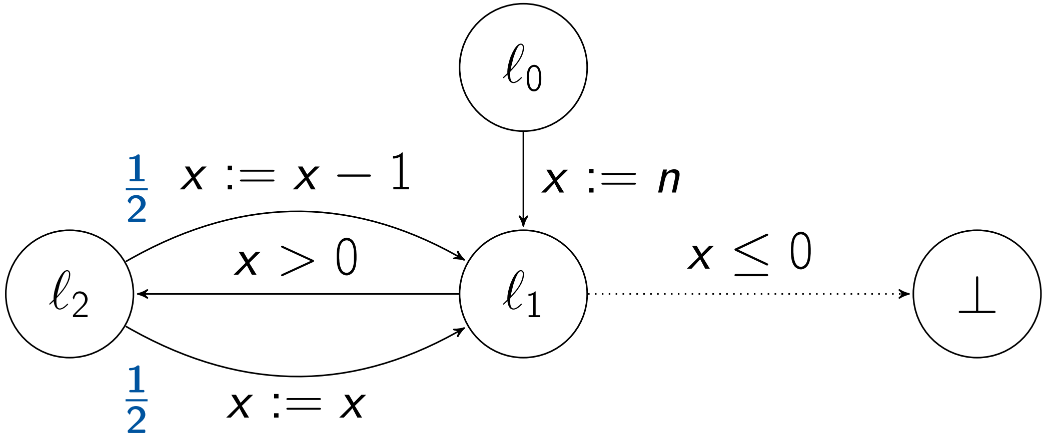
  if $\color{red}{prob(0.5)}$ **then**

  $x := x - 1$

  else

  $\color{red}{x := x}$

$\bot$



Semantics of Probabilistic Programs

Probabilistic Control Flow Graph

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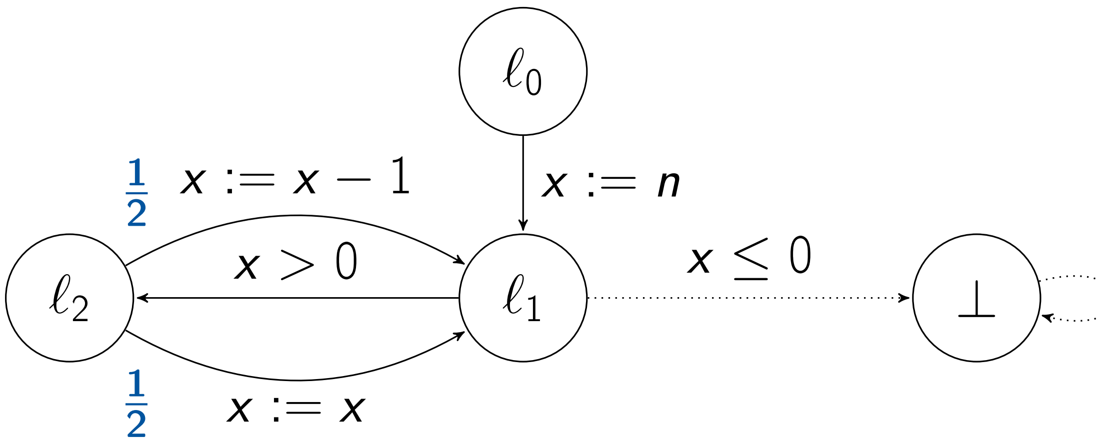
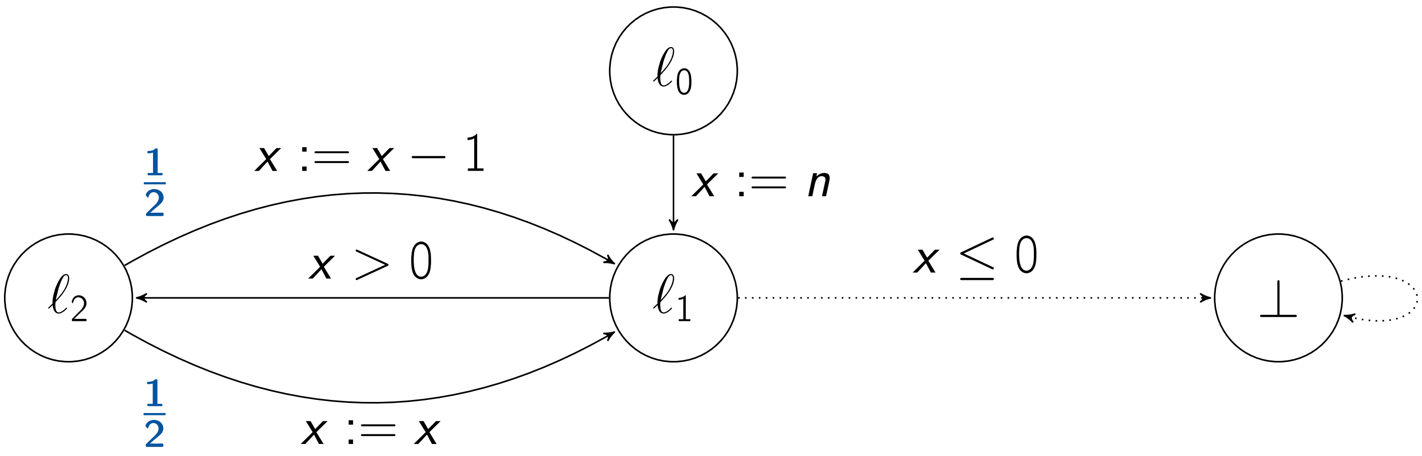
$\bot$

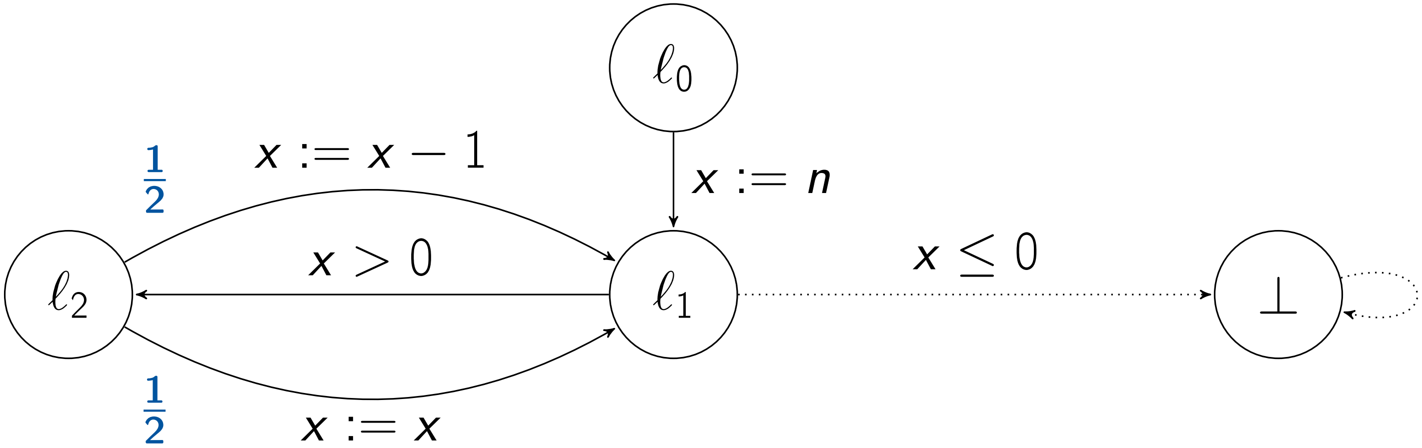


Markov Decision Processes

Markov Decision Processes



Markov Decision Processes

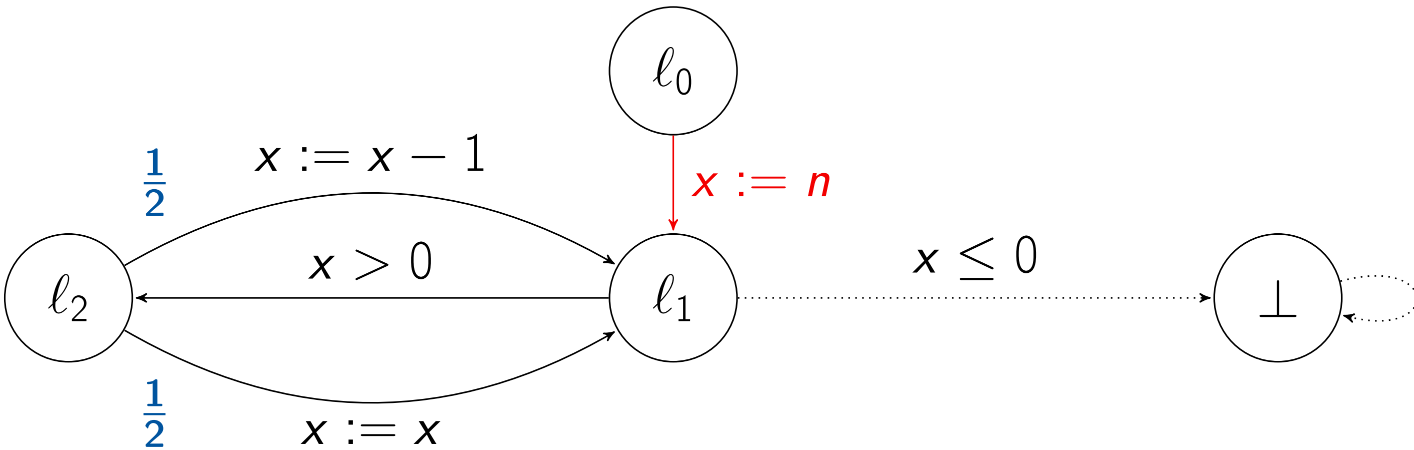


Example

$$(\ell_0, (0, 1))$$

Semantics of Probabilistic Programs

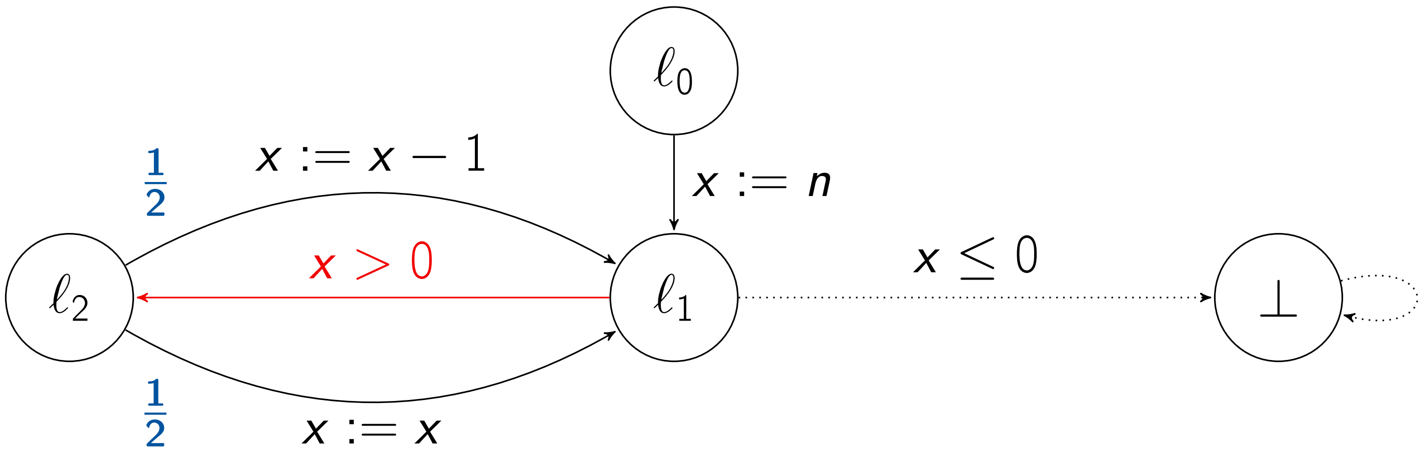
Markov Decision Processes



Example

$$(\ell_0, (0, 1)) \to (\ell_1, (1, 1))$$

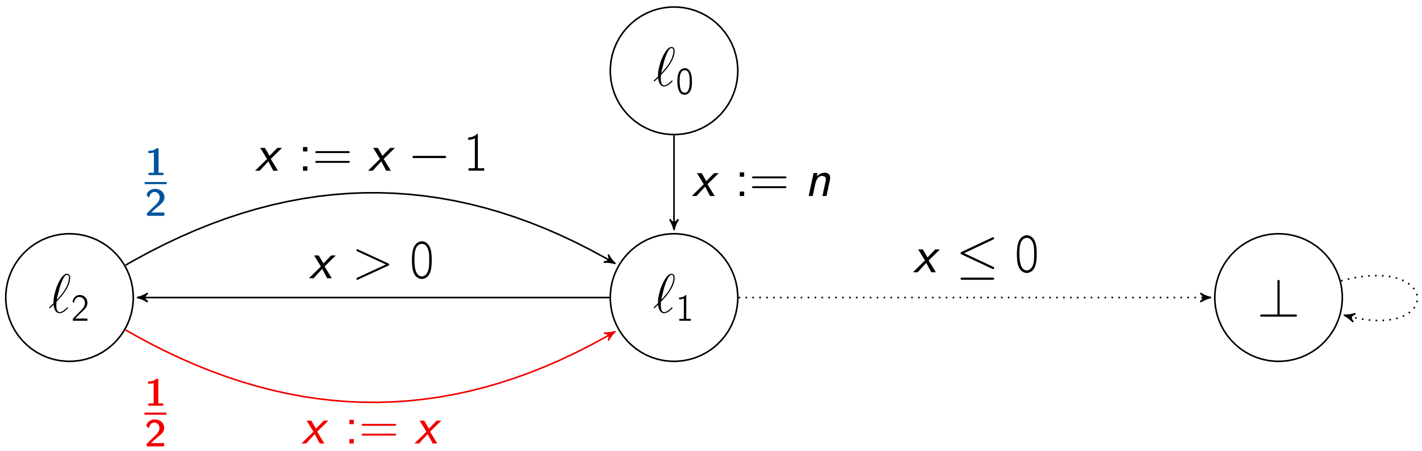
Markov Decision Processes



Example

$$(\ell_0, (0, 1)) \rightarrow (\ell_1, (1, 1)) \rightarrow (\ell_2, (1, 1))$$

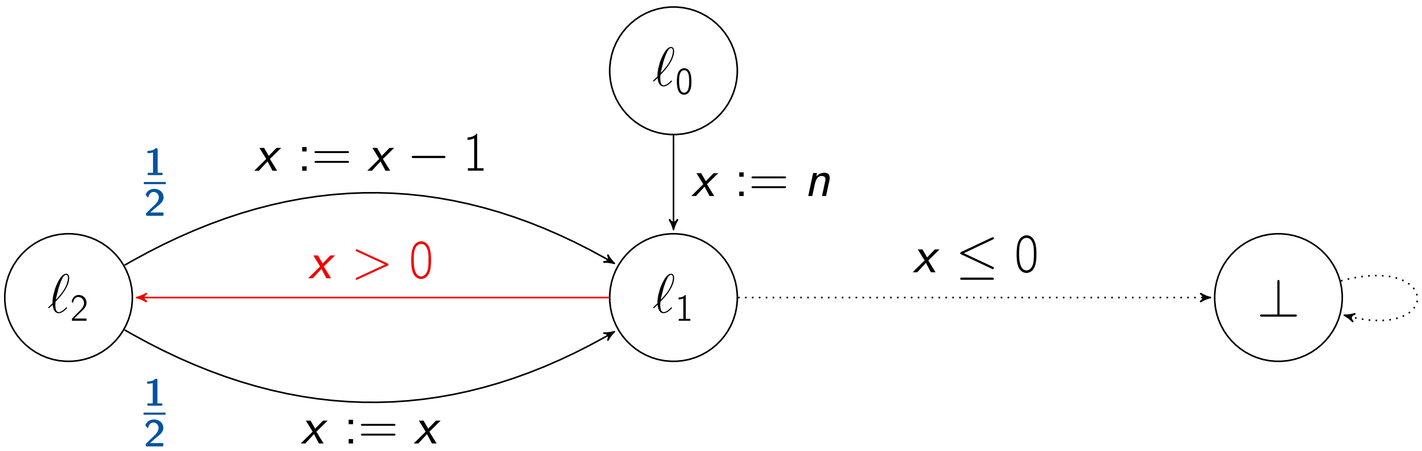
Markov Decision Processes



Example

$$(\ell_0, (0, 1)) \rightarrow (\ell_1, (1, 1)) \rightarrow (\ell_2, (1, 1)) \xrightarrow{\frac{1}{2}} (\ell_1, (1, 1))$$

Markov Decision Processes



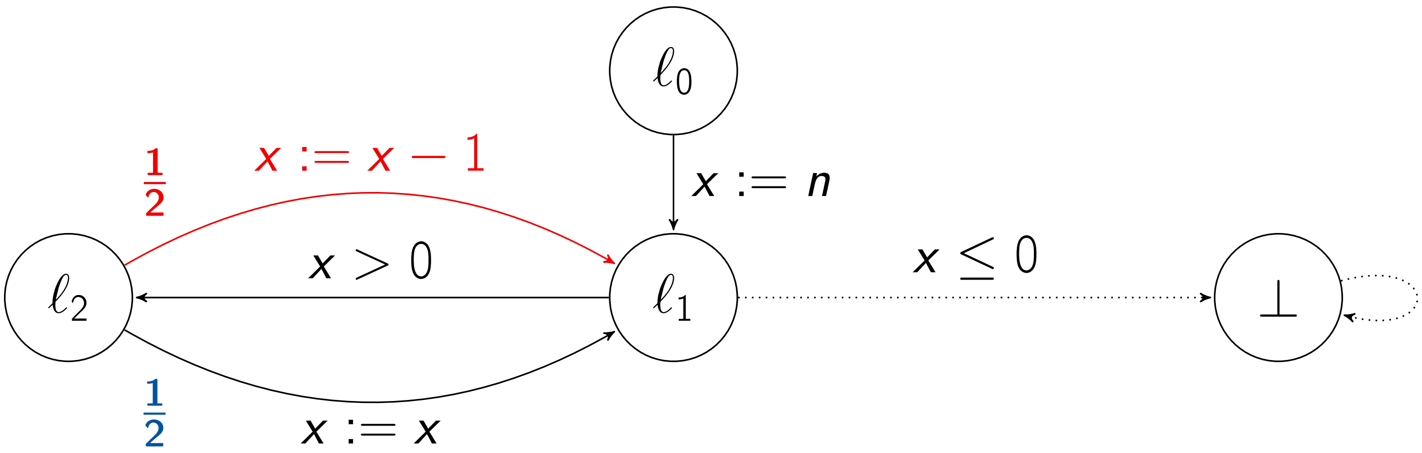
Example

$$(\ell_0, (0,1)) \rightarrow (\ell_1, (1,1)) \rightarrow (\ell_2, (1,1)) \xrightarrow{\frac{1}{2}} (\ell_1, (1,1))$$

$$\rightarrow (\ell_2, (1,1))$$

Semantics of Probabilistic Programs

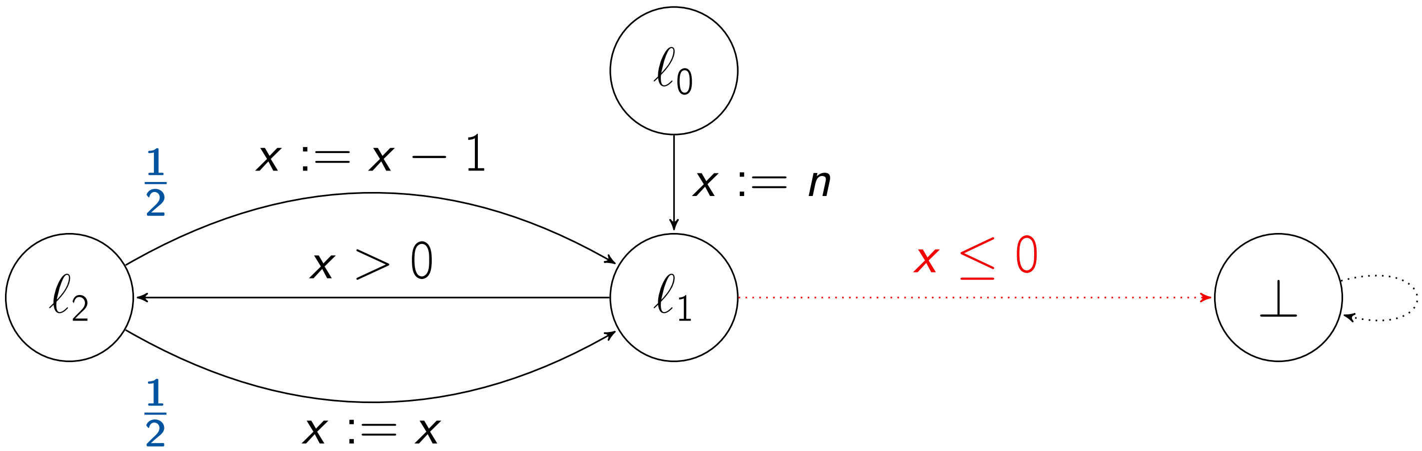
Markov Decision Processes



Example

$$
\begin{aligned}
(\ell_0, (0,1)) \rightarrow (\ell_1, (1,1)) \rightarrow & (\ell_2, (1,1)) \xrightarrow{\frac{1}{2}} (\ell_1, (1,1)) \\
\rightarrow & (\ell_2, (1,1)) \xrightarrow{\frac{1}{2}} (\ell_1, (0,1))
\end{aligned}
$$

Markov Decision Processes



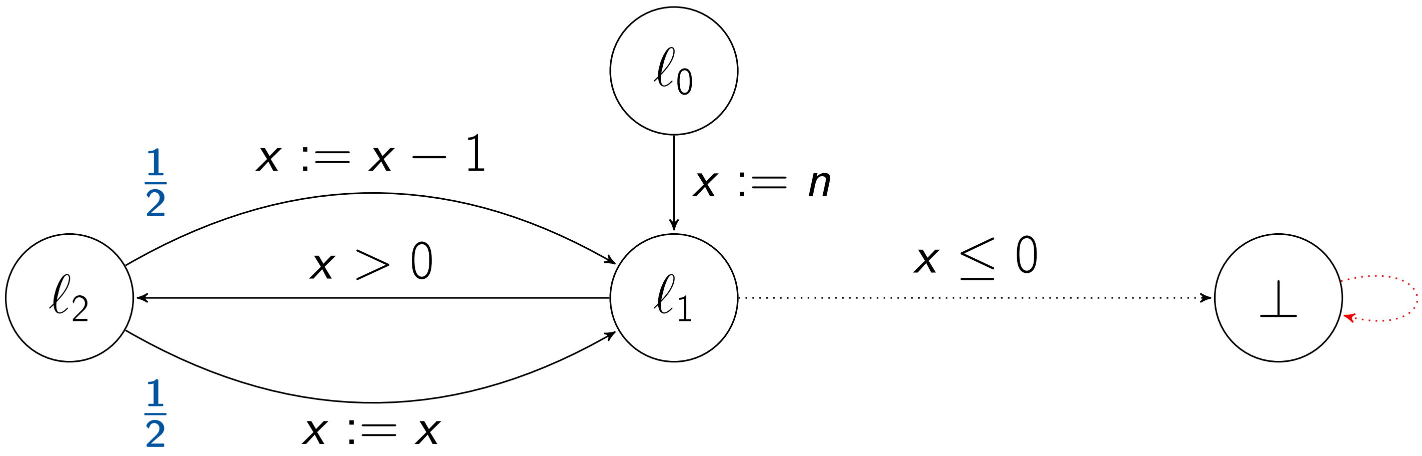
Example

$$(\ell_0, (0,1)) \rightarrow (\ell_1, (1,1)) \rightarrow (\ell_2, (1,1)) \xrightarrow{\frac{1}{2}} (\ell_1, (1,1))$$

$$\rightarrow (\ell_2, (1,1)) \xrightarrow{\frac{1}{2}} (\ell_1, (0,1)) \rightarrow (\bot, (0,1))$$

Semantics of Probabilistic Programs

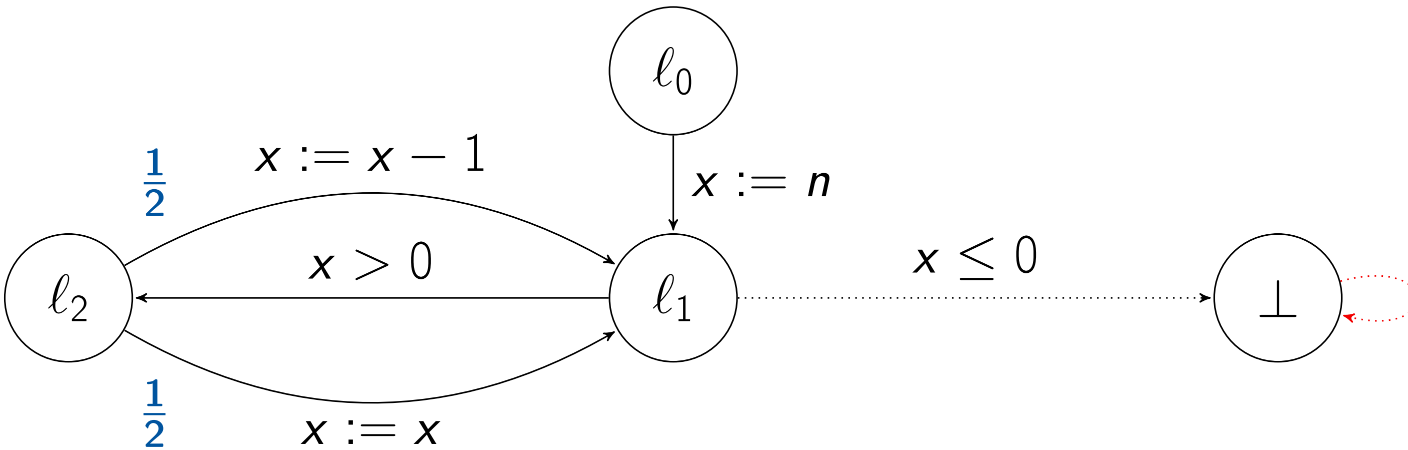
Markov Decision Processes



Example

$$
\begin{aligned}
(\ell_0, (0,1)) \rightarrow (\ell_1, (1,1)) &\rightarrow (\ell_2, (1,1)) \xrightarrow{\frac{1}{2}} (\ell_1, (1,1)) \\
&\rightarrow (\ell_2, (1,1)) \xrightarrow{\frac{1}{2}} (\ell_1, (0,1)) \rightarrow (\perp, (0,1)) \\
&\rightarrow (\perp, (0,1))
\end{aligned}
$$

Markov Decision Processes



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&\rightarrow (\bot, (0,1)) \rightarrow (\bot, (0,1))
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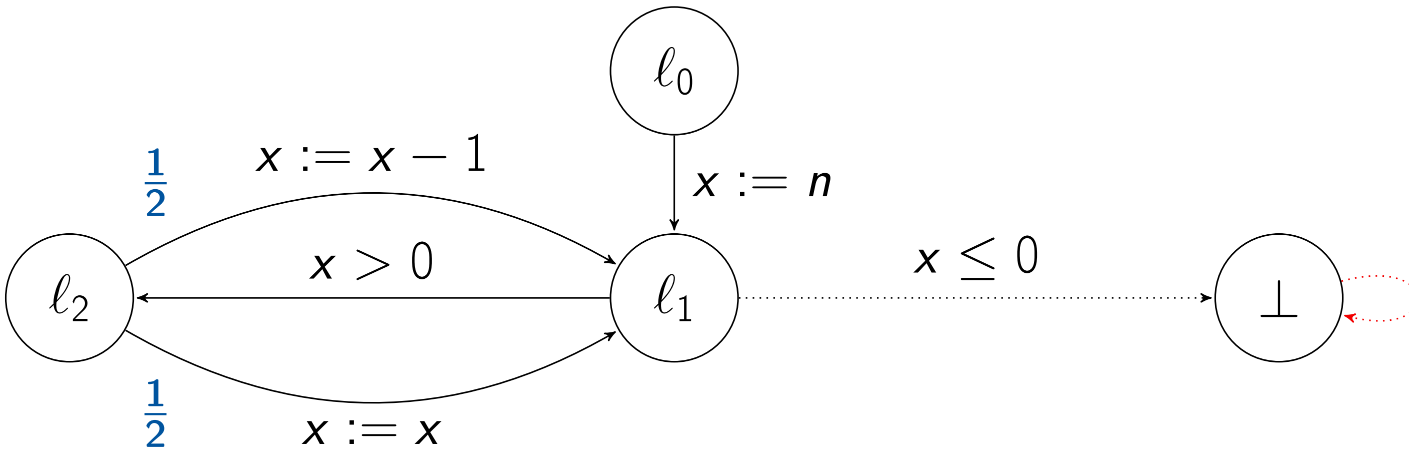
Semantics of Probabilistic Programs

Markov Decision Processes



Example

$$(\ell_0, (0,1)) \to (\ell_1, (1,1)) \to (\ell_2, (1,1)) \xrightarrow{\frac{1}{2}} (\ell_1, (1,1))$$
$$\to (\ell_2, (1,1)) \xrightarrow{\frac{1}{2}} (\ell_1, (0,1)) \to (\bot, (0,1))$$
$$\to (\bot, (0,1)) \to (\bot, (0,1)) \to \ldots$$

Semantics of Probabilistic Programs

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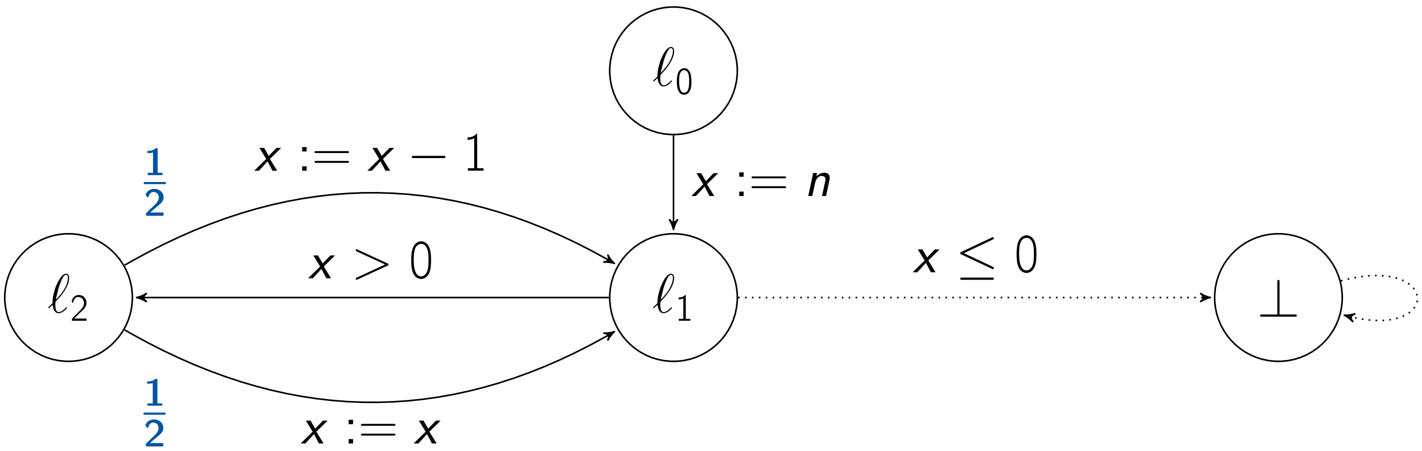
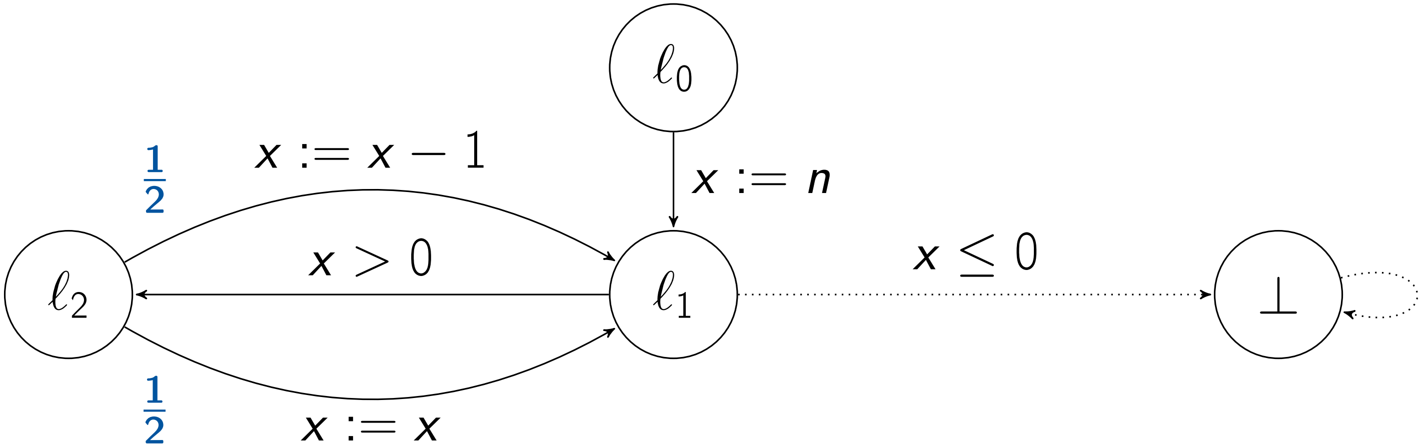


Example

$$\mathbb{P}(\{(\ell_0,(0,1)) \to (\ell_1,(1,1)) \to (\ell_2,(1,1)) \xrightarrow{\frac{1}{2}} (\ell_1,(1,1)) \to (\ell_2,(1,1)) \xrightarrow{\frac{1}{2}} (\ell_1,(0,1)) \to (\bot,(0,1)) \to (\bot,(0,1)) \to (\bot,(0,1)) \to \ldots\})$$

Semantics of Probabilistic Programs

Markov Decision Processes



Example

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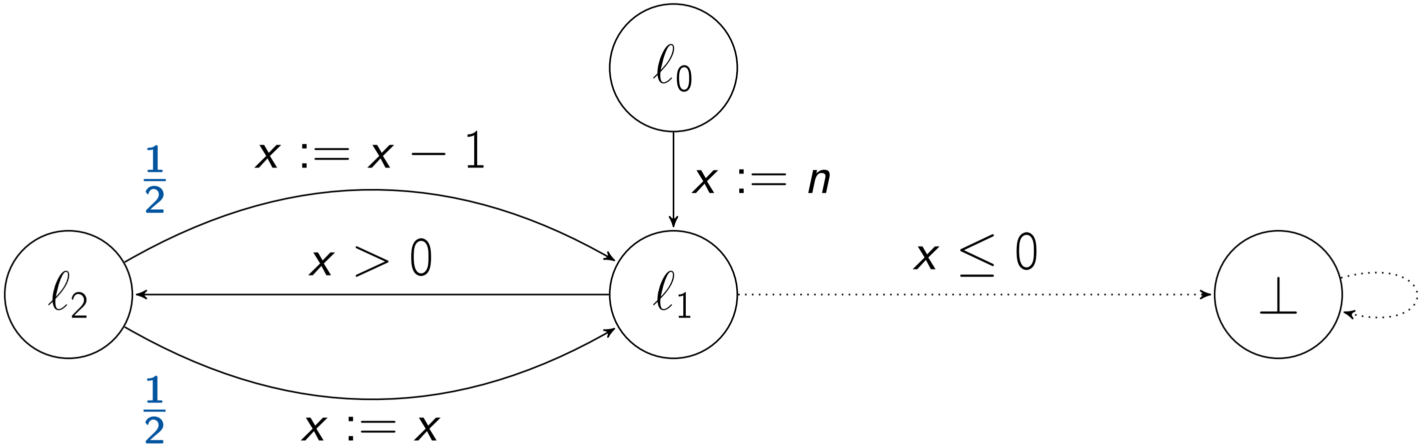
$$\to (\ell_2,(1,1)) \xrightarrow{\frac{1}{2}} (\ell_1,(0,1)) \to (\perp,(0,1))$$
$$\to (\perp,(0,1)) \to (\perp,(0,1)) \to \ldots\}) = \frac{1}{2}\cdot\frac{1}{2}$$

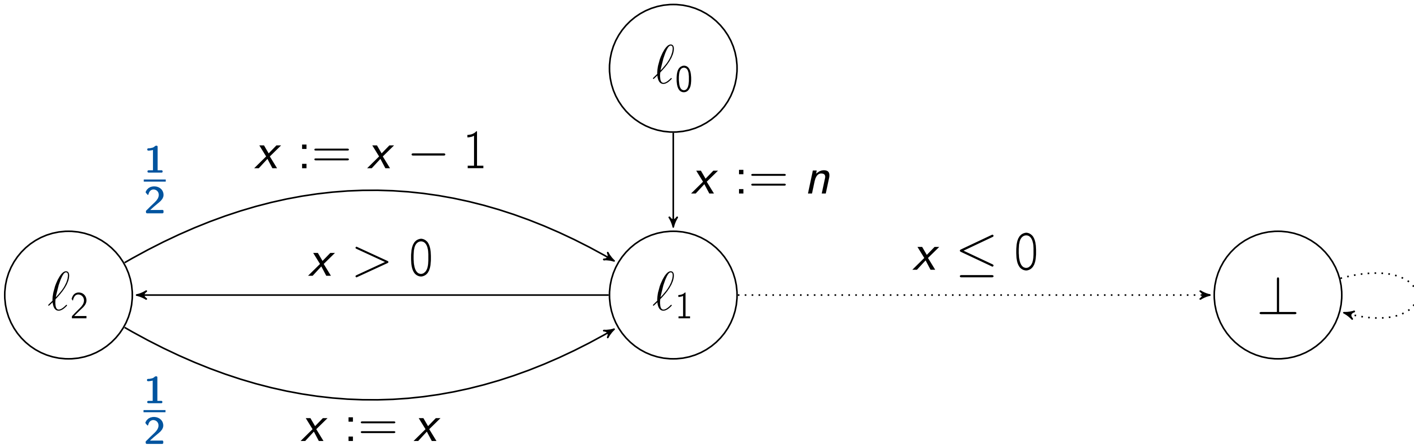
Markov Decision Processes



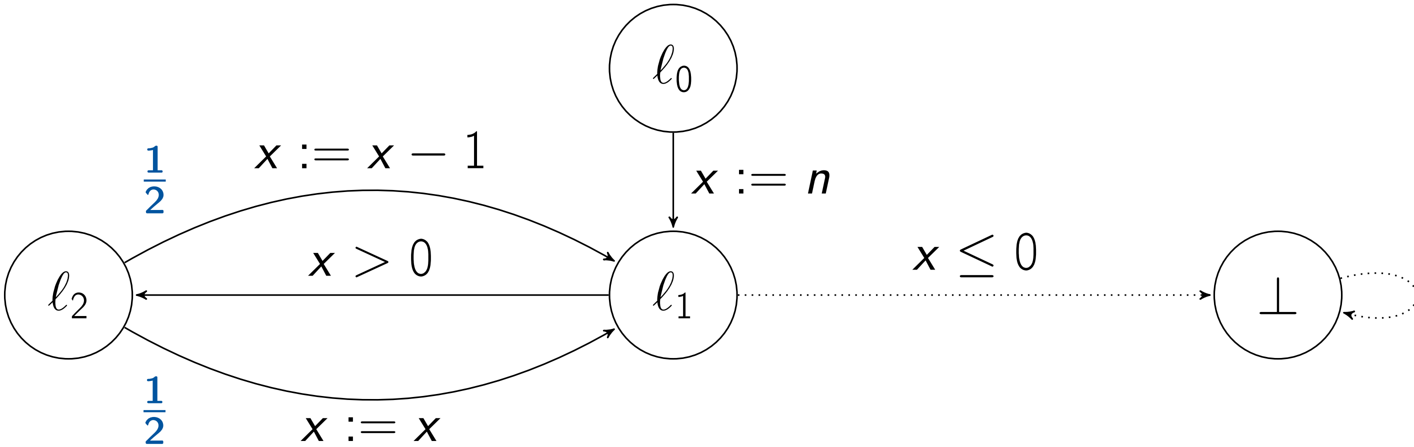
Example

$$\mathbb{P}(\{(\ell_0, (0,1)) \rightarrow (\ell_1, (1,1)) \rightarrow (\ell_2, (1,1)) \xrightarrow{\frac{1}{2}} (\ell_1, (1,1)) \rightarrow (\ell_2, (1,1)) \xrightarrow{\frac{1}{2}} (\ell_1, (0,1)) \rightarrow (\bot, (0,1)) \rightarrow (\bot, (0,1)) \rightarrow (\bot, (0,1)) \rightarrow \ldots\}) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Markov Decision Processes



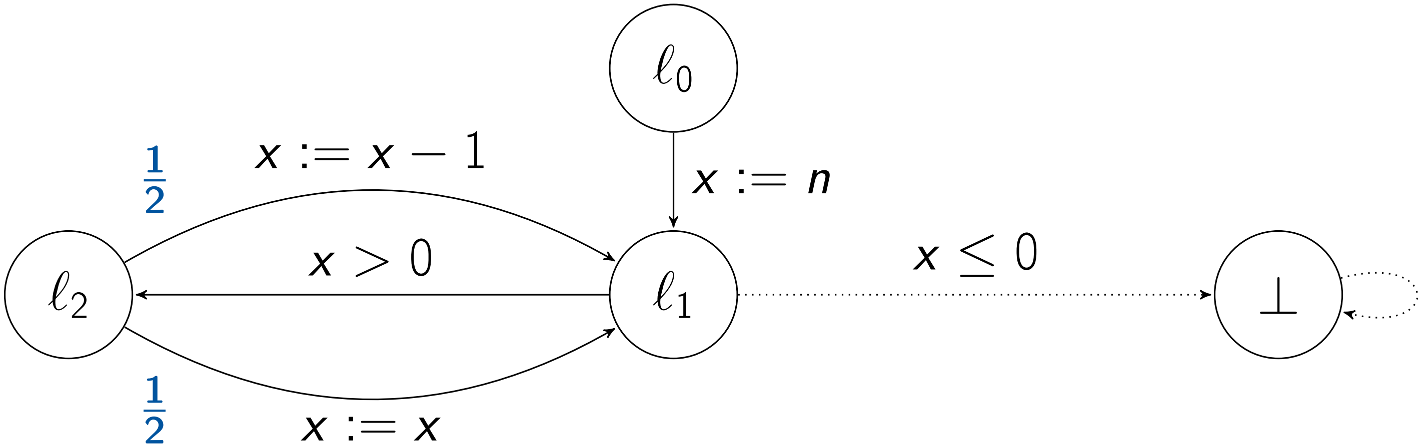
Markov Decision Processes



Definition (Underlying Process)

Semantics of Probabilistic Programs

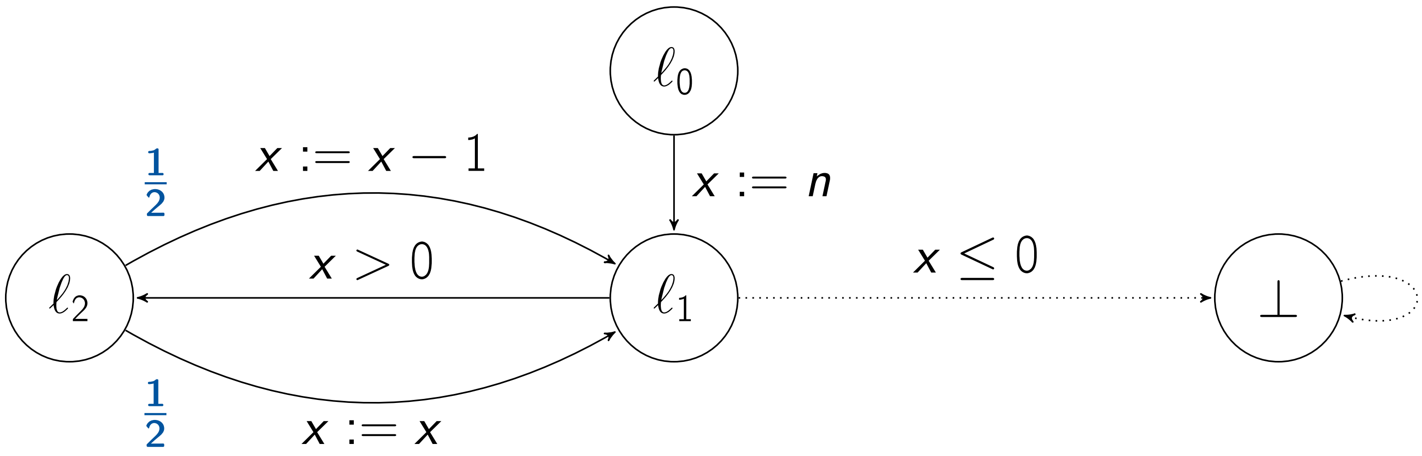
Markov Decision Processes



Definition (Underlying Process)

- $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.

Markov Decision Processes



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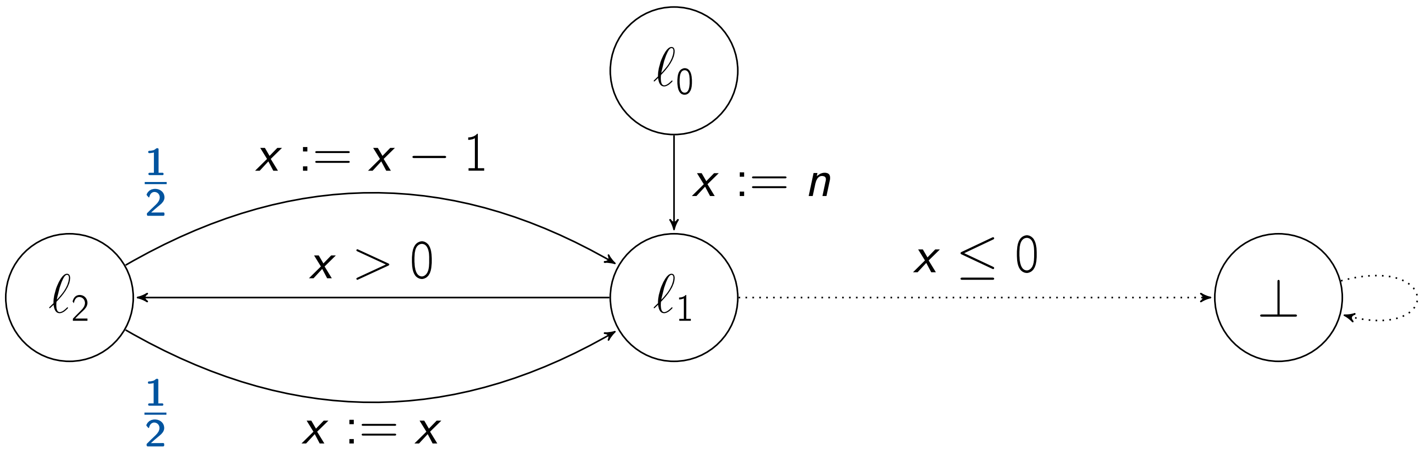
- $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space.
- $\Omega := \{\omega \mid \omega \text{ is a run}\} \subseteq \left(\mathbb{N} \to \mathcal{L} \times \mathbb{Z}^2\right)$,

Semantics of Probabilistic Programs

Markov Decision Processes



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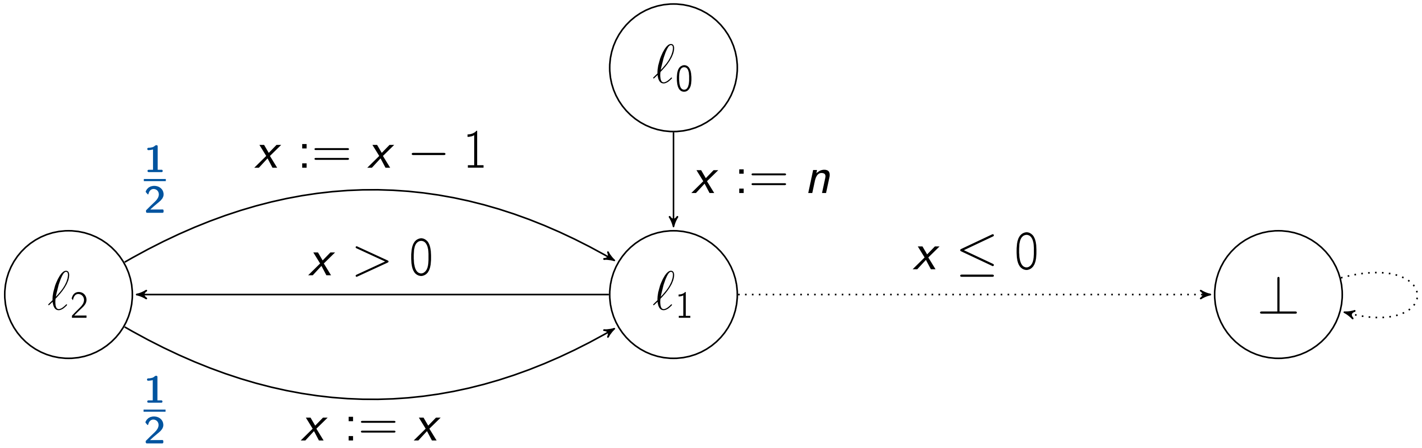
Semantics of Probabilistic Programs

Markov Decision Processes



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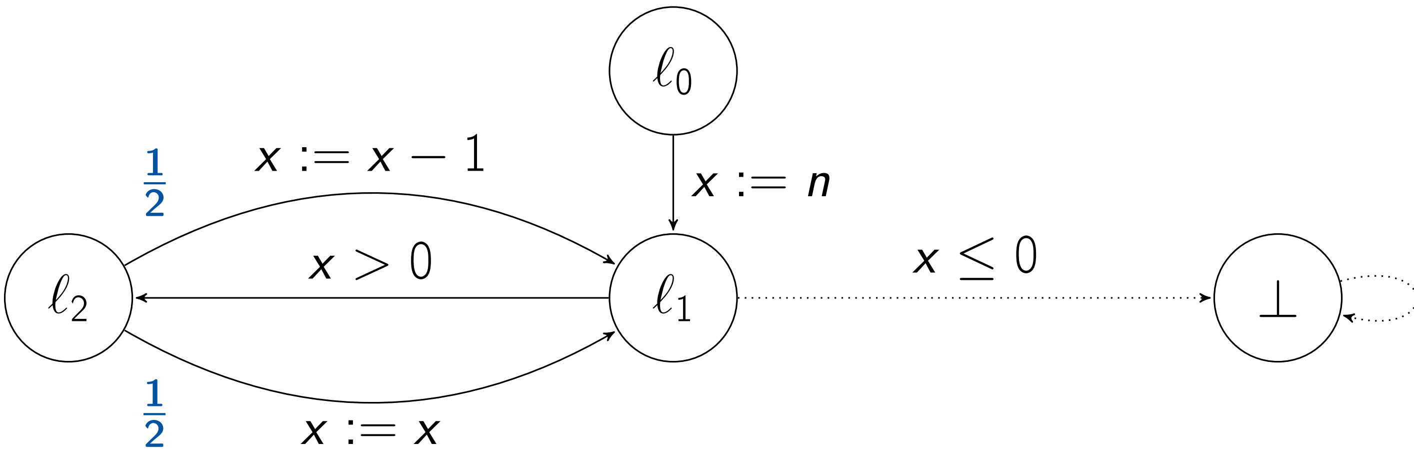
Semantics of Probabilistic Programs

Markov Decision Processes



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- $\mathit{Term} : \Omega \to \mathbb{N}, \omega \mapsto \min\{k \mid \omega_n = (\bot, \dots) \text{ for all } n \geq k\}$.

Markov Decision Process

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Objective

Approximating $\mathbb{E}(\mathit{Term})$ by a stochastic process $(X_i)_{i \in \mathbb{N}}$ with „nice“ properties.

Markov Decision Process

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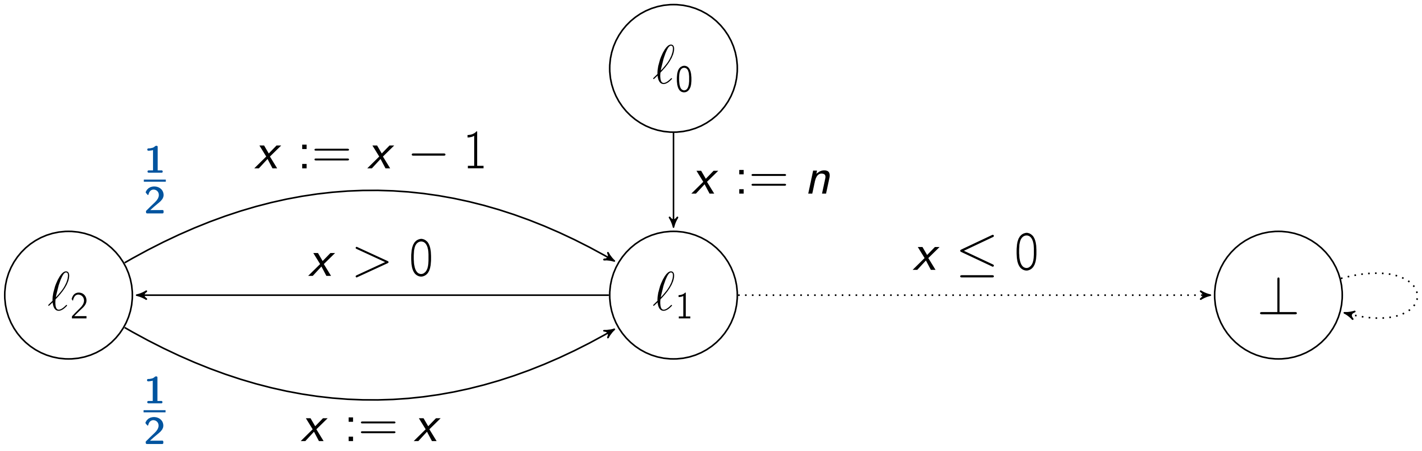
Objective

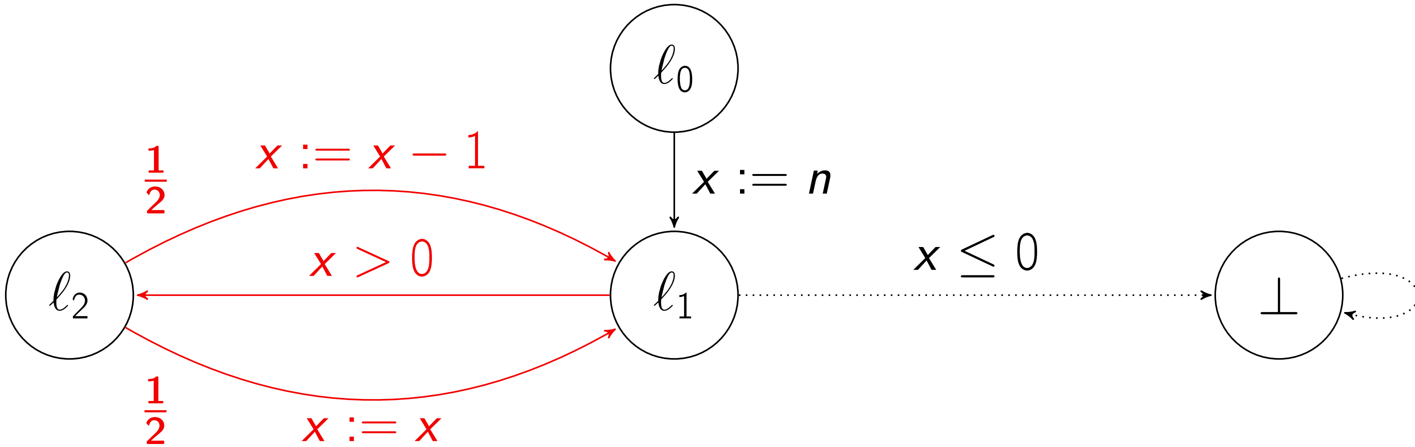
Approximating $\mathbb{E}(\mathit{Term})$ by a stochastic process $(X_i)_{i \in \mathbb{N}}$ with „nice" properties.

„nice" $\leftrightarrow$ decreasing in expectation.

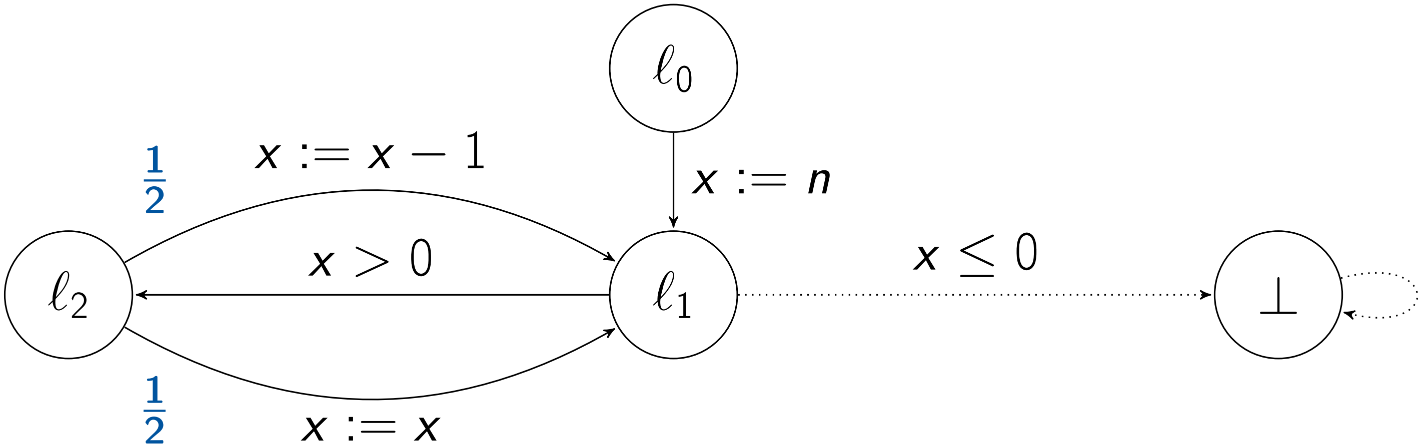
Probabilistic Ranking Functions



Probabilistic Ranking Functions

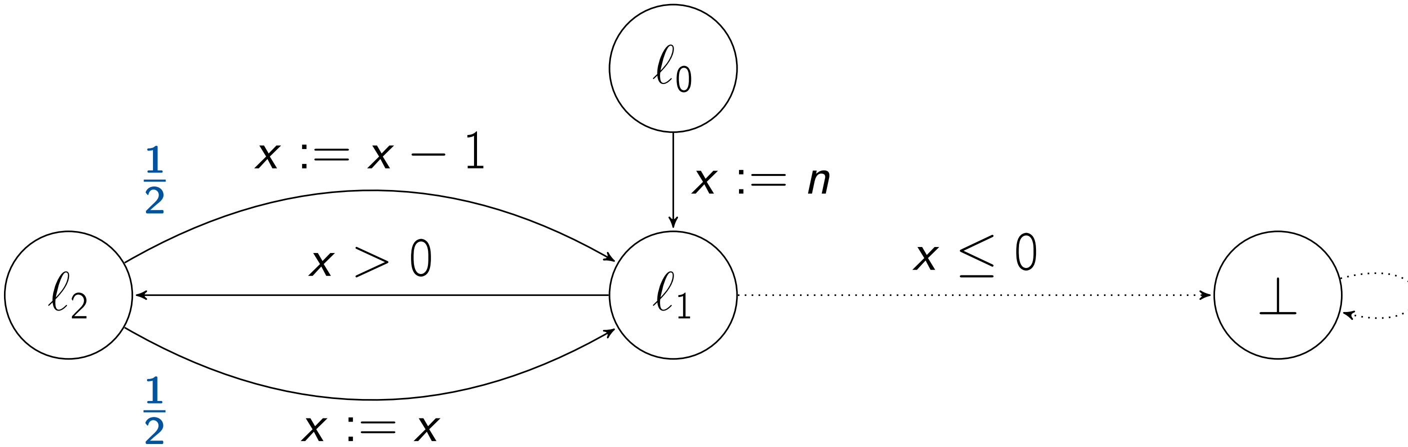


Probabilistic Ranking Functions



Example

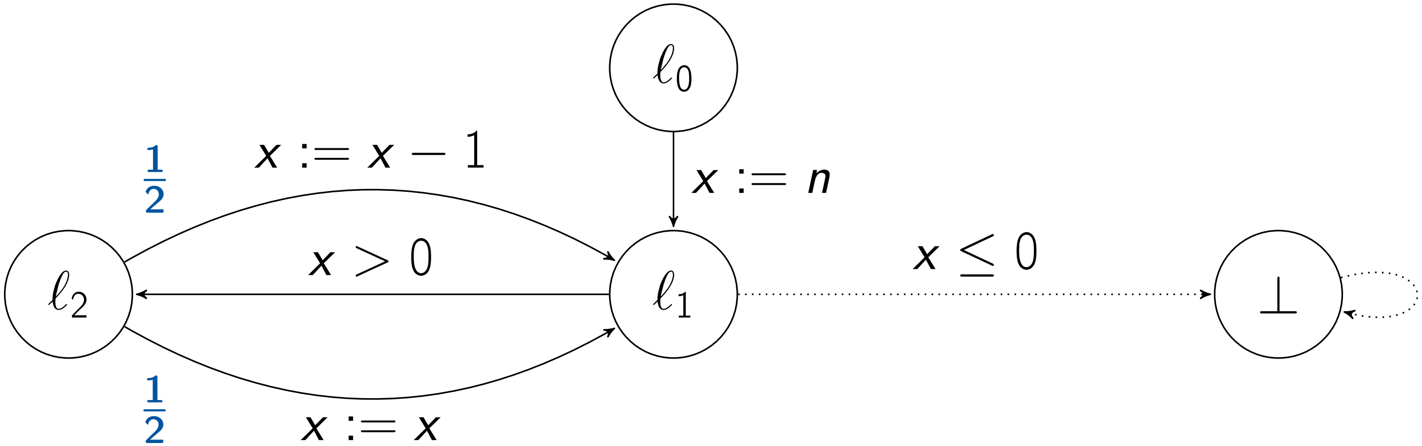
Probabilistic Ranking Functions



Example

$$R : \mathbb{Z}^2 \to \mathbb{Z}, (x, n) \mapsto 2 \cdot x$$

Probabilistic Ranking Functions

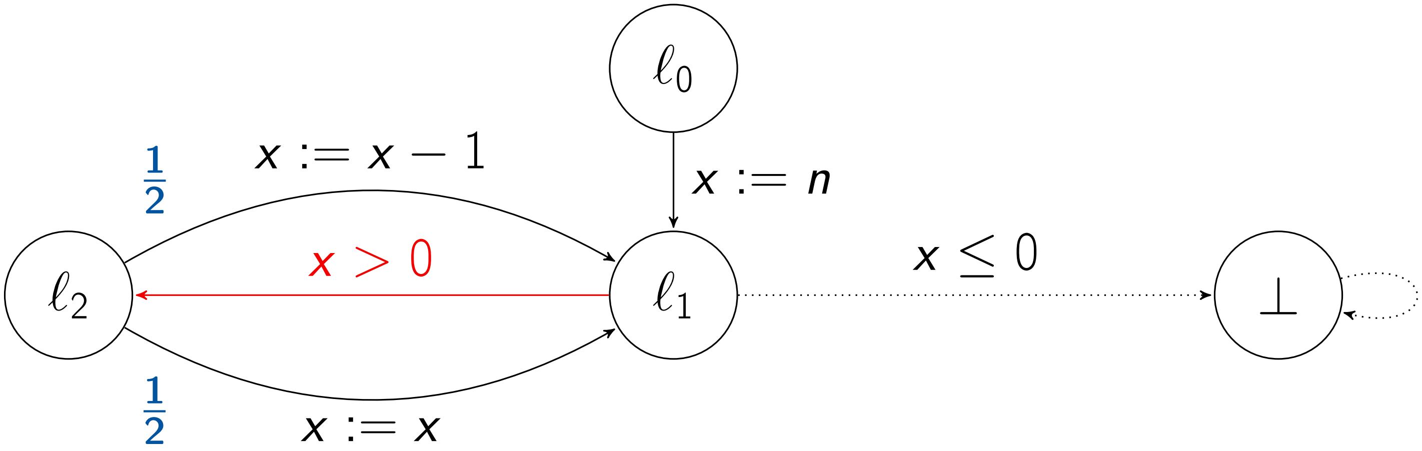


Example

$$R : \mathbb{Z}^2 \to \mathbb{Z}, (x, n) \mapsto 2 \cdot x$$

$$R((x, n)) = 2 \cdot x$$

Probabilistic Ranking Functions



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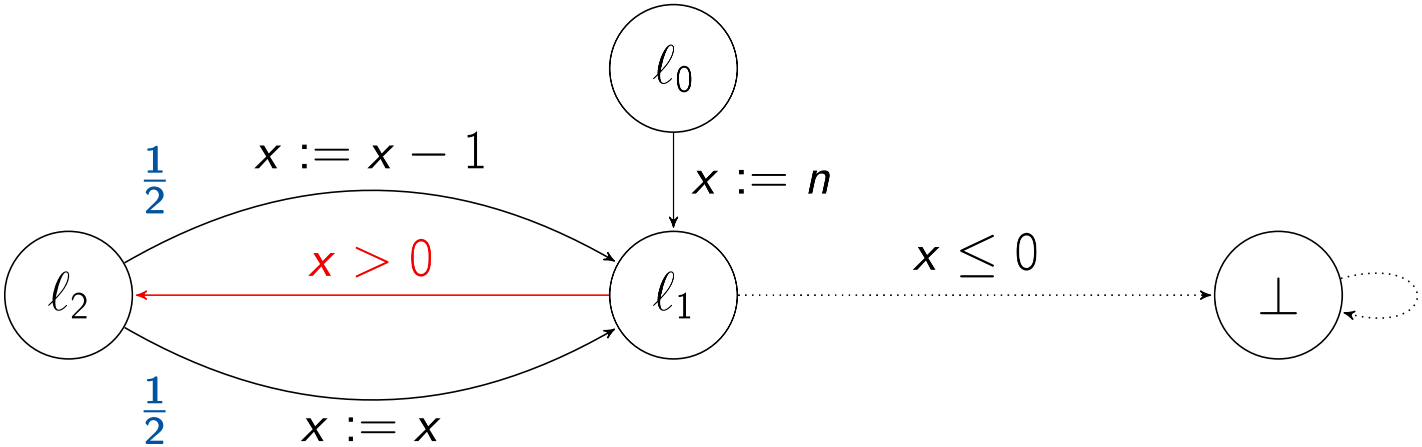

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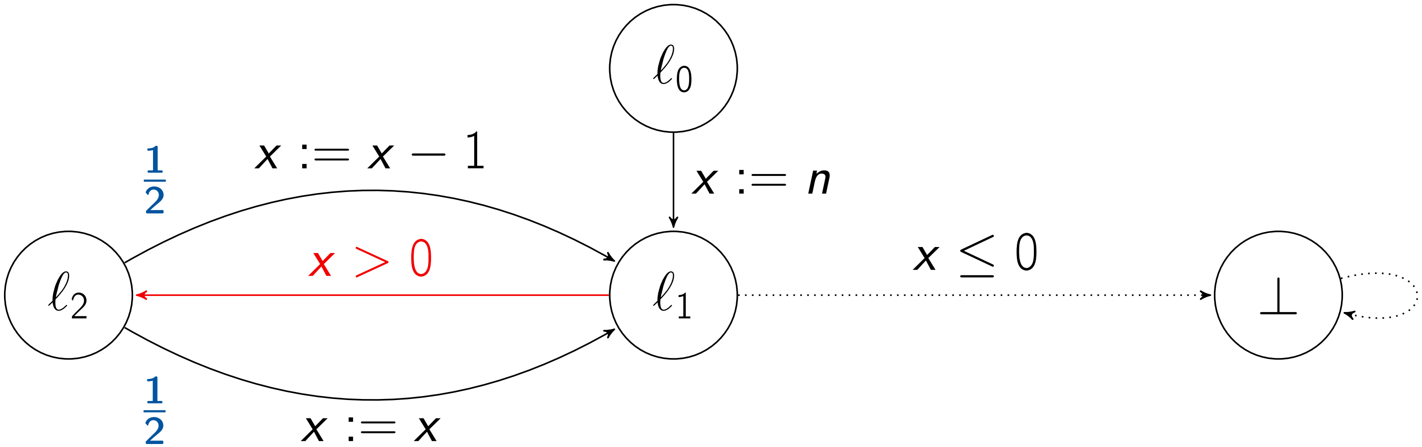
Probabilistic Ranking Functions

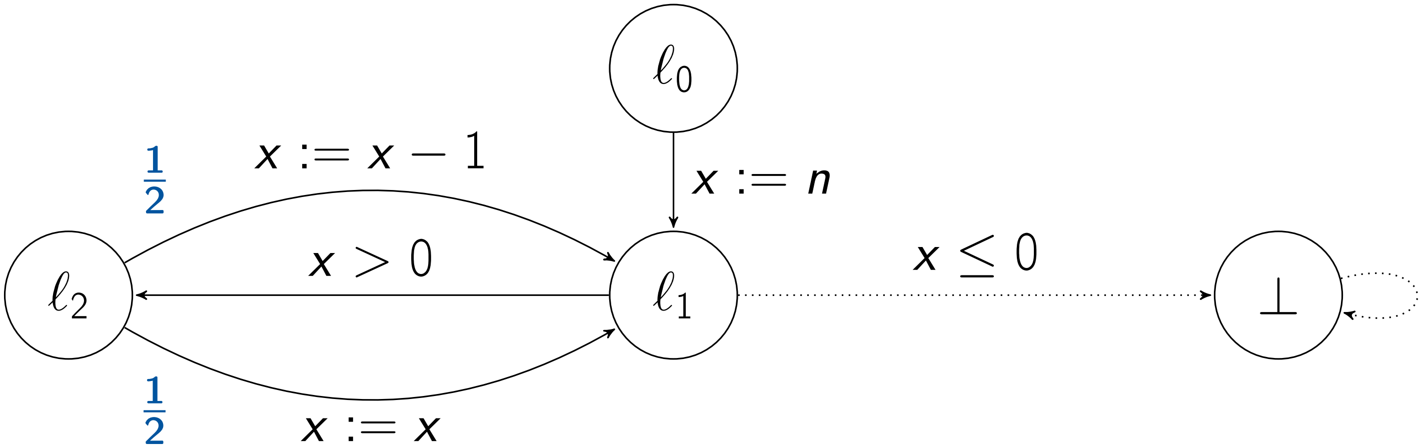


Example

$$R : \mathbb{Z}^2 \to \mathbb{Z}, (x, n) \mapsto 2 \cdot x$$

$$R((x, n)) = 2 \cdot x \geq 0 \Rightarrow \mathbb{E}(R)(x, n) \geq 0$$

Probabilistic Ranking Functions



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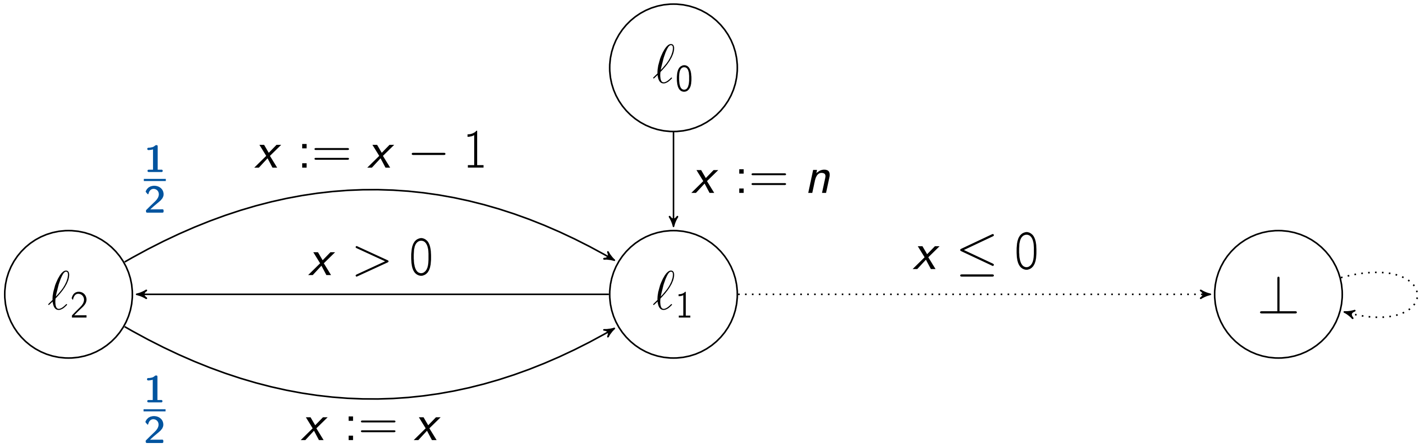

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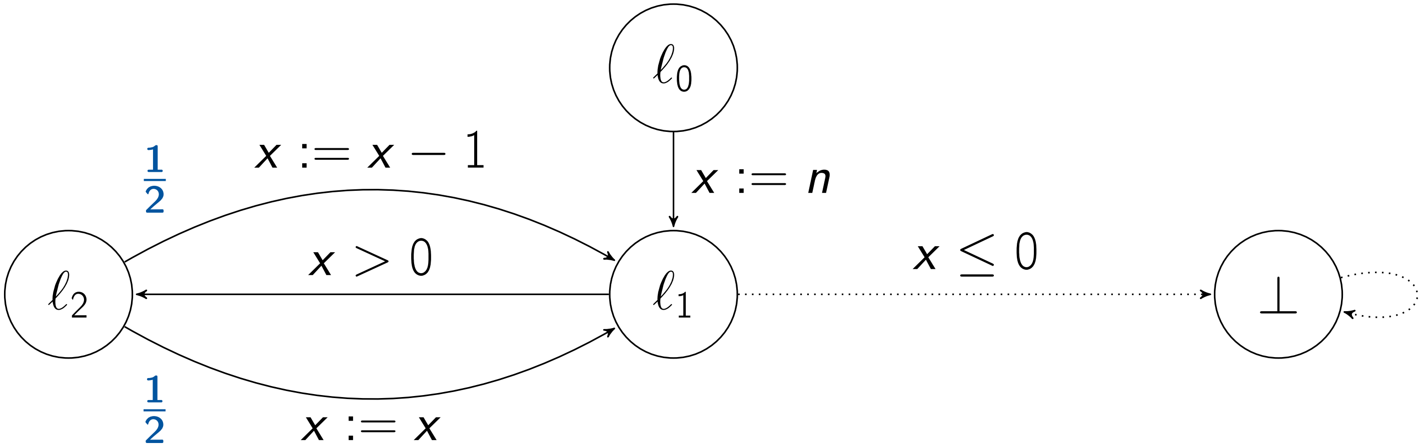

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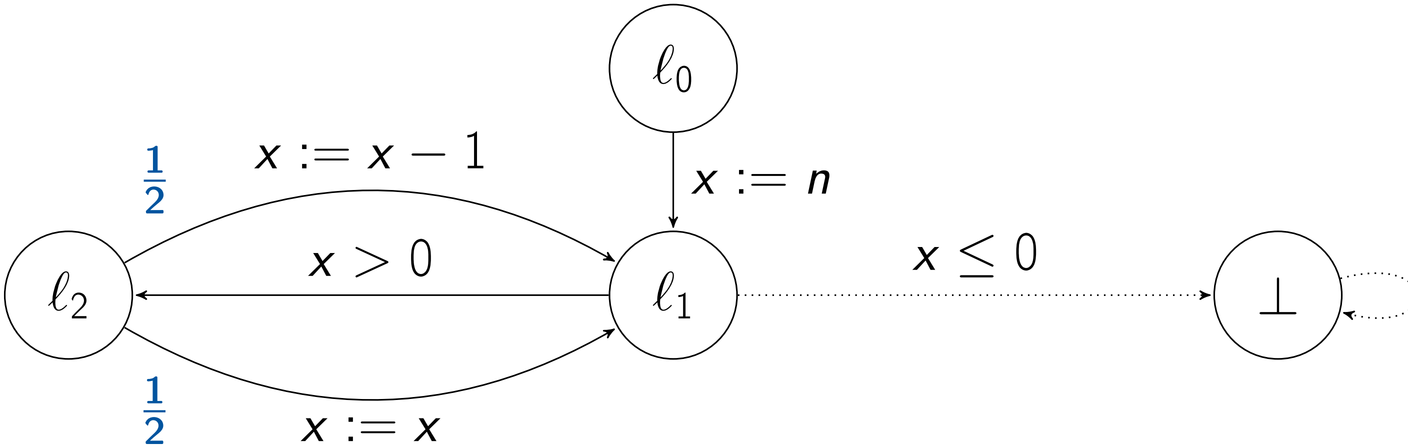
$$R : \mathbb{Z}^2 \to \mathbb{Z}, (x, n) \mapsto 2 \cdot x$$

$R((x, n)) = 2 \cdot x > 2 \cdot x - 1$

Probabilistic Ranking Functions



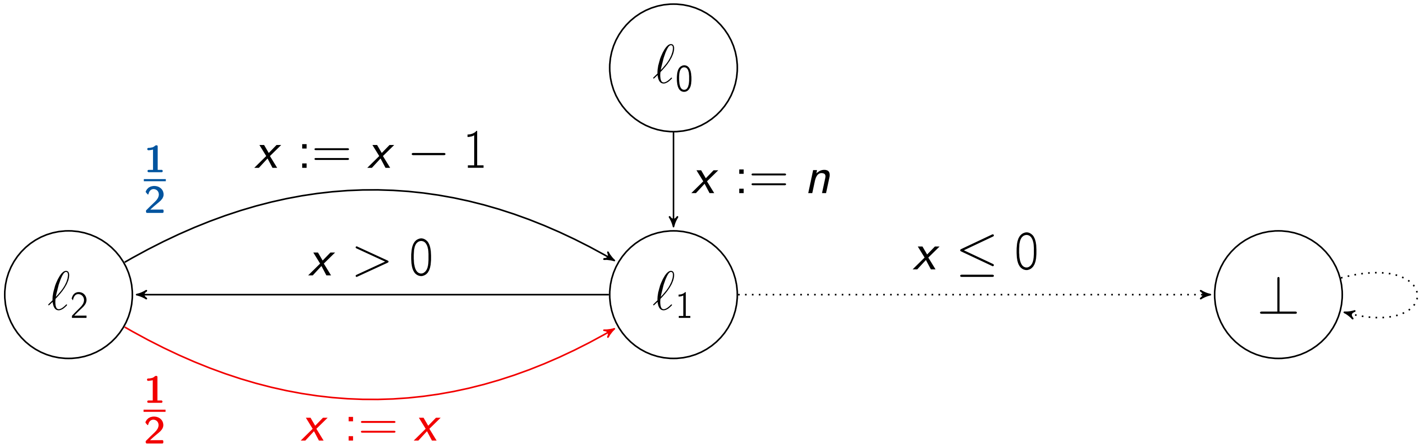
Example

$$R : \mathbb{Z}^2 \to \mathbb{Z}, (x, n) \mapsto 2 \cdot x$$

$$R((x, n)) = 2 \cdot x > 2 \cdot x - 1 = x + x - 1$$

Semantics of Probabilistic Programs

Probabilistic Ranking Functions



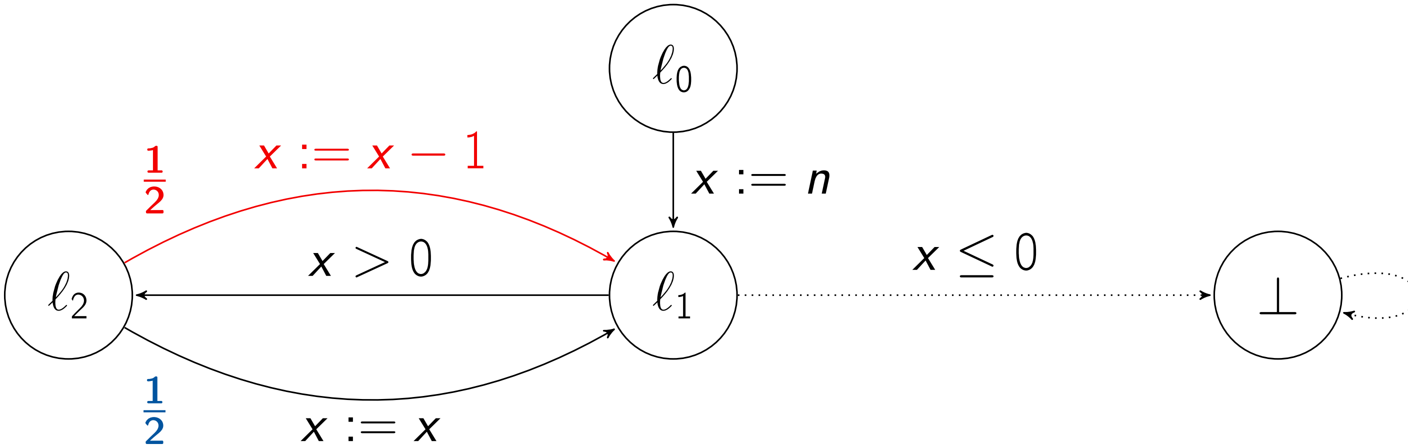
Example

$$R : \mathbb{Z}^2 \to \mathbb{Z}, (x, n) \mapsto 2 \cdot x$$

$$R((x, n)) = 2 \cdot x > 2 \cdot x - 1 = x + x - 1 = \frac{1}{2} R(x)$$

Semantics of Probabilistic Programs

Probabilistic Ranking Functions



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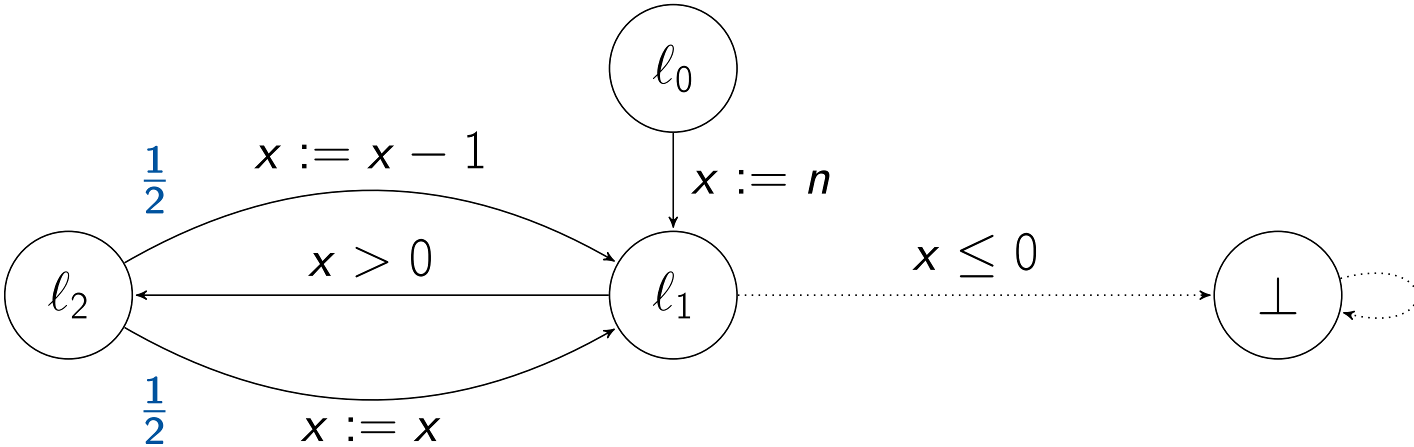
Probabilistic Ranking Functions



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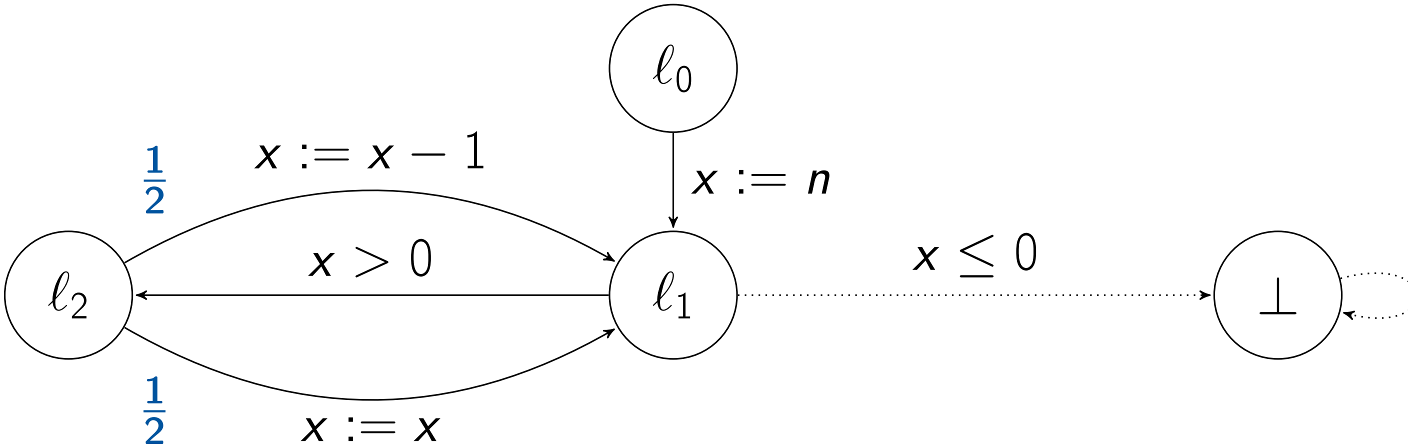
Probabilistic Ranking Functions

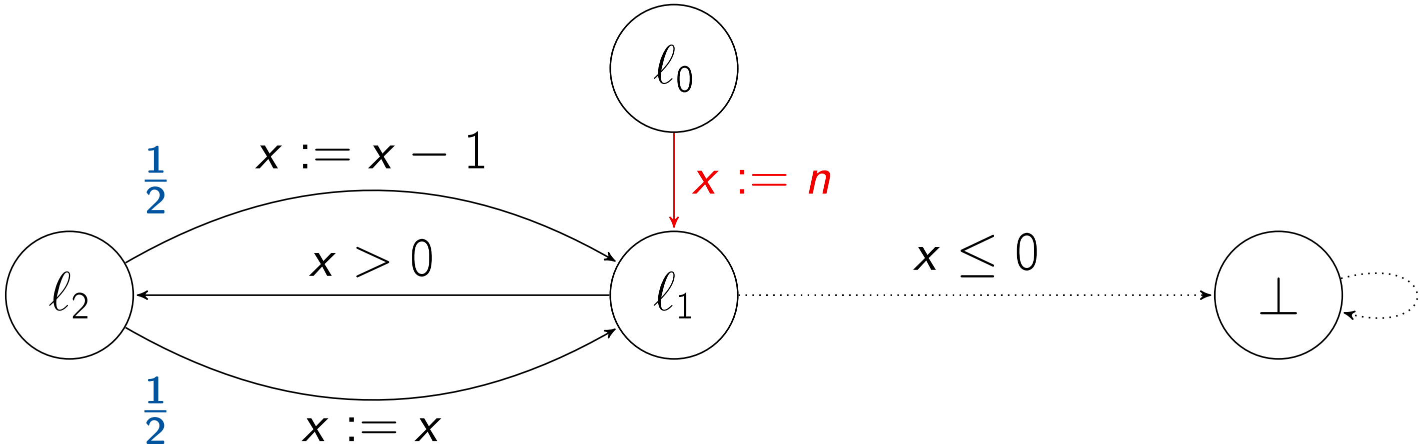


Example

$$R : \mathbb{Z}^2 \to \mathbb{Z}, (x, n) \mapsto 2 \cdot x$$

$$\mathbb{E}(\mathit{Term})$$

Probabilistic Ranking Functions

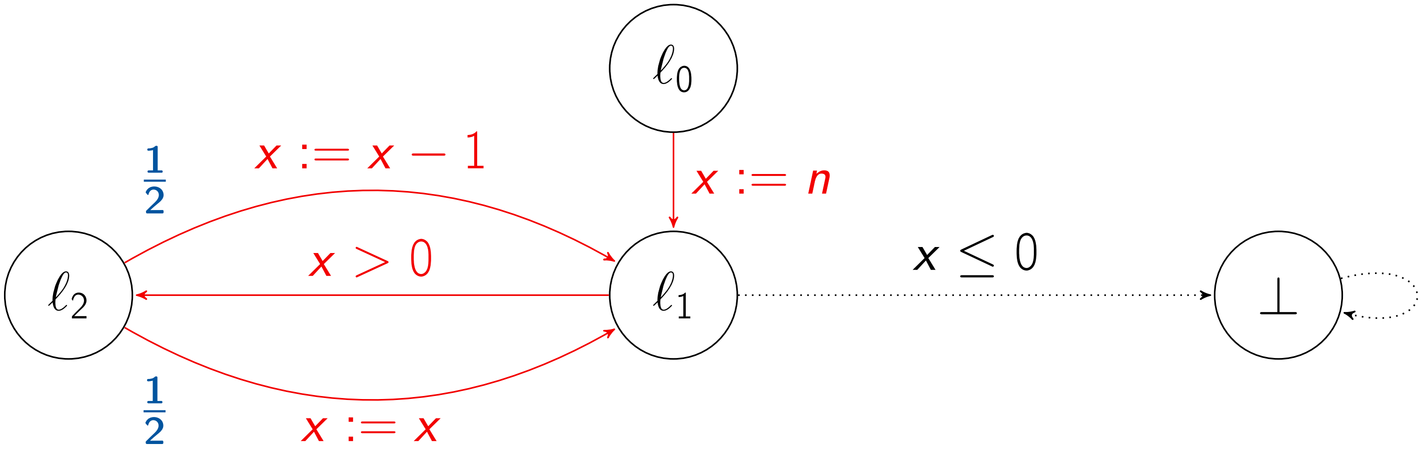


Example

$$R : \mathbb{Z}^2 \to \mathbb{Z}, (x, n) \mapsto 2 \cdot x$$

$$\mathbb{E}(\mathit{Term}) \leq 1$$

Probabilistic Ranking Functions

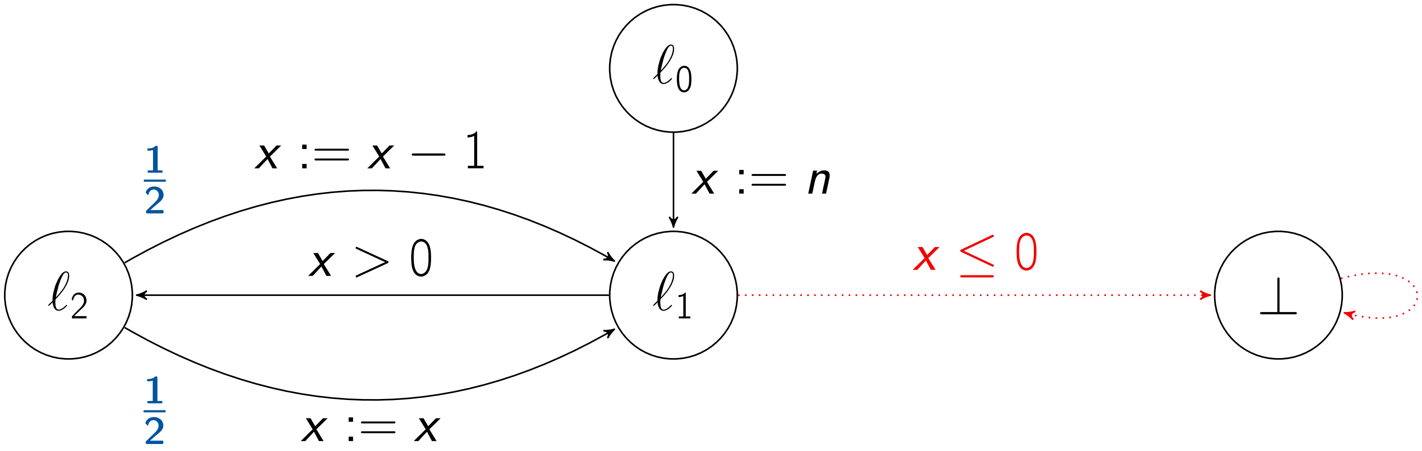


Example

$$R : \mathbb{Z}^2 \to \mathbb{Z}, (x, n) \mapsto 2 \cdot x$$

$$\mathbb{E}(\mathit{Term}) \leq 1 + \mathbb{E}(R)(n, n)$$

Probabilistic Ranking Functions

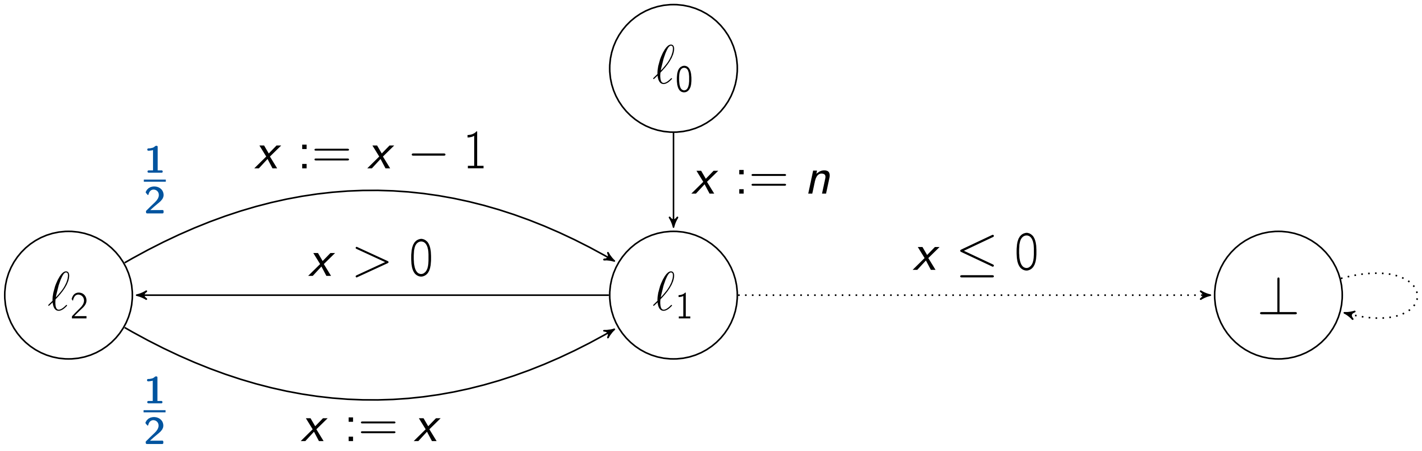


Example

$$R : \mathbb{Z}^2 \to \mathbb{Z}, (x, n) \mapsto 2 \cdot x$$

$$\mathbb{E}(\mathit{Term}) \leq 1 + \mathbb{E}(R)(n, n) + 0$$

Probabilistic Ranking Functions

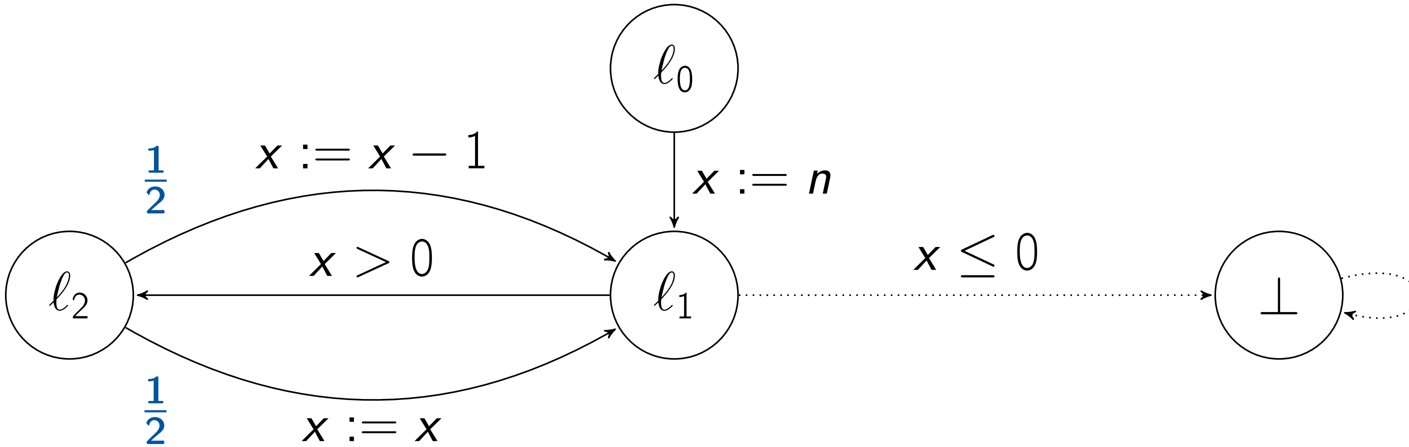


Example

$$R : \mathbb{Z}^2 \to \mathbb{Z}, (x, n) \mapsto 2 \cdot x$$

$$\mathbb{E}(\mathit{Term}) \leq 1 + \mathbb{E}(R)(n, n) + 0 = 2n + 1$$

Probabilistic Ranking Functions



Example

$$R : \mathbb{Z}^2 \to \mathbb{Z}, (x, n) \mapsto 2 \cdot x$$

R can be generated automatically.

Outline

Modularity with Expected Size

Example

$x := n$

while $x > 0$ **do**

$\quad x := \frac{1}{2}\langle x-1\rangle + \frac{1}{2}\langle x\rangle$

My Work

Modularity with Expected Size

Example

$x := n$

$y = 0$

while $x > 0$ do

$\quad y := y + x$

$\quad x := \frac{1}{2}\langle x - 1\rangle + \frac{1}{2}\langle x\rangle$

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$\rightarrow \mathbb{E}(y) \in \mathcal{O}(n^2)$

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while $y > 0$ do

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$\rightarrow$ Expectation of y after the first loop is at most quadratic in n.

My Work

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$x := n$

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$\rightarrow \mathbb{E}(y) \in \mathcal{O}(n^2)$

while $y > 0$ **do**

$\quad y := y - 1$

$\rightarrow$ Expectation of y after the first loop is at most quadratic in n.

$\rightarrow$ Expected runtime of the second loop is linear in y.

My Work

Modularity with Expected Size

Example

$x := n$

$y = 0$

while $x > 0$ **do**

| $y := y + x$

| $x := \frac{1}{2}\langle x - 1\rangle + \frac{1}{2}\langle x\rangle$

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while $y > 0$ **do**

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$\rightarrow$ Expectation of y after the first loop is at most quadratic in n.

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$\rightarrow$ Expected runtime of the program is at most quadratic.

My Work

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while $y > 0$ **do**

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→ Expectation of y after the first loop is at most quadratic in n.

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→ Modular approach in „Koat“ [ACM–TOPLAS 2016] uses size-approximations.

My Work

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$\rightarrow$ Modular approach in „Koat“ [ACM–TOPLAS 2016] uses size-approximations.

$\rightarrow$ Expected size is needed for generalising the modular approach.

My Work

Lower bounds

My Work

Lower bounds

Example

$x := 1$
$k := 0$
while $x = 1$ **do**
  $x := \frac{1}{2}\langle 0 \rangle + \frac{1}{2}\langle x \rangle$
  $k := k + 1$

My Work

Lower bounds

Example

$x := 1$
$k := 0$
while $x = 1$ **do**
  $x := \frac{1}{2}\langle 0 \rangle + \frac{1}{2}\langle x \rangle$
  $k := k + 1$

$\rightarrow$ $\mathbb{E}(\mathit{Term})$ can be constructed as a **least fixpoint** of a certain function.

My Work

Lower bounds

Example

$x := 1$
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while $x = 1$ **do**
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My Work

Lower bounds

Example

$x := 1$
$k := 0$
while $x = 1$ **do**
  $x := \frac{1}{2}\langle 0 \rangle + \frac{1}{2}\langle x \rangle$
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$\rightarrow$ $\mathbb{E}(\mathit{Term})$ can be constructed as a least fixpoint of a certain function.

$\rightarrow$ Underapproximation of $\mathbb{E}(\mathit{Term})$ is more difficult than overapproximation.

$\rightarrow$ Important to infer tight bounds.

Conclusion

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- Hot research topic at the moment.

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- Applications in analysing stochastic processes.

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- Hot research topic at the moment.
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- Lower bounds have to be considered.

References

Thank You

[1] Sheshansh Agrawal, Krishnendu Chatterjee, and Petr Novotný. Lexicographic ranking supermartingales: an efficient approach to termination of probabilistic programs. *POPL*, 2018.

[2] Olivier Bournez and Florent Garnier. Proving positive almost sure termination. *RTA*, 2005.

[3] Benjamin Lucien Kaminski and Joost-Pieter Katoen. On the hardness of almost–sure termination. *MFCS*, 2015.

[4] Benjamin Lucien Kaminski, Joost-Pieter Katoen, Christoph Matheja, and Federico Olmedo. Weakest precondition reasoning for expected run-times of probabilistic programs. *ESOP*, 2016.