Provenance Analysis for Guarded Logics

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joint work with Erich Grädel

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UnRAVeL | RWTH AACHEN UNIVERSITY

Structure

Introduction to Provenance

Modal Logic and the Guarded Fragment of FO

Model Checking Games

Complexity

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Motivation for Provenance in Logic

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  - relevance of subformulae for the truth of the whole

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- Get information from that valuation like for instance
  - number of distinct proofs
  - relevance of subformulae for the truth of the whole
- Evaluate formulae under certain aspects like
  - access control
  - confidence

Basic Idea

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Examples:

- The natural numbers $\mathbb{N} = (\mathbb{N}, +, \cdot, 0, 1)$.
- The boolean semiring $\mathbb{B} = (\{\top, \bot\}, \vee, \wedge, \top, \bot)$.
- The Viterbi semiring $\mathbb{V} = ([0, 1], \max, \cdot, 0, 1)$.
- The semiring $\mathbb{N}[X] = (\mathbb{N}[X], +, \cdot, 0, 1)$ of multivariate polynomials over $\mathbb{N}$ with variables in X.

Semiring Valuations for FO

Definition. Let K be a semiring and $\mathfrak{A}$ a τ-structure. A consistent K-interpretation is a function $\pi \colon \mathrm{Lit}_A(\tau) \to K$ such that $\pi(\alpha(\bar{a})) \cdot \pi(\neg\alpha(\bar{a})) = 0$ for all atomic α and all $\bar{a} \in A^k$.

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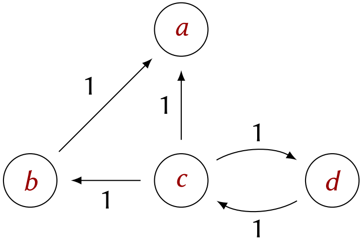
Formula: $\forall x \exists y (Exy \lor Eyx)$



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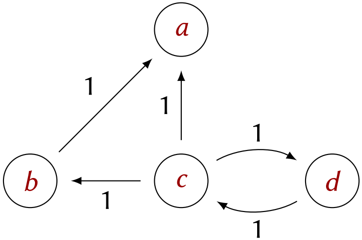

$$\prod_{x \in A} \sum_{y \in A} (\pi(Exy) + \pi(Eyx))$$
$$= (\pi(Eba) + \pi(Eca)) \cdot (\pi(Eba) + \pi(Ecb)) \cdot$$
$$(\pi(Eca) + \pi(Ecb) + \pi(Ecd) + \pi(Edc)) \cdot (\pi(Edc) + \pi(Ecd))$$
$$= 2 \cdot 2 \cdot 4 \cdot 2 = 32$$

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Guarded logics usually have nice algorithmic properties. For the fragments of FO for instance

- decidable satisfiability (in PSPACE for ML, in 2EXPTIME for GF)
- model checking in PTIME (in contrast to PSPACE for FO)
- finite model property

Modal Logic

Syntax:

$$\varphi ::= \bot \mid \top \mid P_i \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \neg\varphi \mid \Diamond\varphi \mid \Box\varphi$$

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Semantics over a Kripke structure (or transition system) $\mathcal{K} = (V, E, (P_i)_{i \in I})$:

- $\mathcal{K}, v \models \Diamond\varphi \quad \Leftrightarrow \quad$ there is a $w \in vE$ such that $\mathcal{K}, w \models \varphi$
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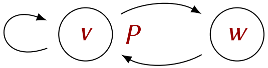

$\mathcal{K}, v \models \Diamond P$ but $\mathcal{K}, v \not\models \Box P$

Provenance for Modal Logic

A modal *K*-interpretation $\pi : (\{P_i \mid i \in I\} \cup \{\neg P_i \mid i \in I\}) \times V \to K$ with an edge evaluation function $f : E \to K$ extends to a *K*-valuation $\pi : \mathrm{ML} \times V \to K$ by

$$\pi(\bot, v) := 0 \qquad \pi(\top, v) := 1$$

$$\pi(P_i, v) := \pi(P_i, v) \qquad \pi(\neg P_i, v) := \pi(\neg P_i, v)$$

$$\pi(\psi \vee \varphi, v) := \pi(\psi, v) + \pi(\varphi, v) \qquad \pi(\psi \wedge \varphi, v) := \pi(\psi, v) \cdot \pi(\varphi, v)$$

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Consider $\varphi = \Box P$ and $\varphi'(x) = \forall y(Exy \to Py) \equiv \forall y(\neg Exy \vee Py)$.
Then $\mathcal{K}, w \vDash \varphi$ is equivalent to $\mathcal{K} \vDash \varphi'(w)$.

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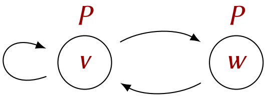

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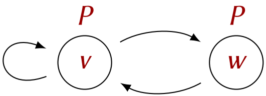

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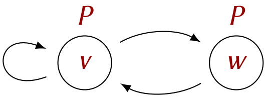

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$$\begin{aligned}\pi(\varphi'(w)) &= \prod_{y \in V} (\pi(\neg Ewy) + \pi(Py)) \\ &= (\pi(\neg Ewv) + 1) \cdot (\pi(\neg Eww) + 1).\end{aligned}$$

The Guarded Fragment of First-Order Logic

Definition (Andréka, van Benthem, Németi). The set of guarded formulae $\mathrm{GF}(\tau)$ has the following syntax:

$$\varphi ::= R\bar{a} \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \neg\varphi \mid (\exists \bar{y}.\alpha)\varphi \mid (\forall \bar{y}.\alpha)\varphi$$

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The semantics is given by

$$\mathfrak{A} \models (\exists \bar{y}.\alpha)\varphi(\bar{y}) \quad \Leftrightarrow \quad \mathfrak{A} \models \varphi(\bar{t}) \text{ for some } \bar{t} \text{ with } \mathfrak{A} \models \alpha(\bar{t})$$

$$\mathfrak{A} \models (\forall \bar{y}.\alpha)\varphi(\bar{y}) \quad \Leftrightarrow \quad \mathfrak{A} \models \varphi(\bar{t}) \text{ for all } \bar{t} \text{ with } \mathfrak{A} \models \alpha(\bar{t})$$

In particular $(\exists \bar{y}.\alpha)\varphi(\bar{y})$ is equivalent to $\exists \bar{y}(\alpha(\bar{y}) \wedge \varphi(\bar{y}))$ and $(\forall \bar{y}.\alpha)\varphi(\bar{y})$ is equivalent to $\forall \bar{y}(\alpha(\bar{y}) \rightarrow \varphi(\bar{y}))$.

Provenance for the Guarded Fragment

A *K*-interpretation $\pi : \mathsf{Lit}_{\mathfrak{A}} \to K$ can be extended to a *K*-valuation $\pi : \mathsf{GF}(\tau) \to K$ by

- $\pi((\exists \bar{y}.\alpha)\varphi(\bar{y})) = \sum_{\mathfrak{A} \models \alpha(\bar{a})} \pi(\alpha(\bar{a})) \cdot \pi(\varphi(\bar{a}))$
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Again $(\forall \bar{y}.\alpha)\varphi(\bar{y})$ in general does not get the same provenance value as the translation $\forall \bar{y}(\alpha(\bar{y}) \to \varphi(\bar{y}))$ into FO.

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The Model Checking Game for GF

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  - from $(\exists \bar{y}.\alpha)\varphi(\bar{y})$ to $\varphi(\bar{a})$ for all $\bar{a}$ with $\mathfrak{A} \models \alpha(\bar{a})$
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- Verifier moves at disjunctions and existential quantifiers, Falsifier at conjunctions and universal quantifiers
- At nodes of the form $(\neg)R(\bar{a})$ Verifier wins if they hold in $\mathfrak{A}$ and falsifier wins otherwise
- A Player who cannot move loses

Model checking games for ML can be defined analogously:

- Positions φ, v with φ subformula of ψ and $v \in V$
- Literals, $\wedge$, $\vee$ same as for GF
- Edges from $\Diamond\varphi, v$ and $\Box\varphi, v$ to φ, w for all $w \in vE$, labeled with Evw
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In contrast, the FO model checking game has size $O(|\psi| \cdot |A|^{\text{width}(\psi)})$.

Provenance for the GF Model Checking Game

Let $f_0 : \mathrm{Lit}_{\mathfrak{A}} \to K$ be a K-interpretation of the literals.
Let F be the set of edges in the model checking game and Q the set of nodes.

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- $f_0(v) = \sum_{w \in vF} h((v, w)) \cdot f_0(w)$ if v belongs to Verifier
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Proposition (Grädel, Tannen). Let $\varphi(\bar{a})$ be any position in the model checking game for ψ on $\mathfrak{A}$. Let f_0 be the K-valuation for the model checking game and π the K-valuation for the formula. Then $f_0(\varphi(\bar{a})) = \pi(\varphi(\bar{a}))$.

Structure

Complexity of Provanace Evaluation for GF and ML

Proposition (Grädel, D.). Given a formula $\psi \in \mathsf{ML}$ or $\psi \in \mathsf{GF}$ and a corresponding K-interpretation π, the provenance value of ψ can be computed with $O(|\psi| \cdot |\pi|)$ semiring operations.

Complexity of Provanace Evaluation for GF and ML

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$|\ | : K \to \mathbb{N}$ is additively bounded if $|a+b|, |a \cdot b| = O(|a| + |b|)$.
$|\ | : K \to \mathbb{N}$ is multiplicatively bounded if $|a+b|, |a \cdot b| = O(|a| \cdot |b|)$.

Example

A possible cost measure for the semiring of natural numbers $(\mathbb{N}, +, \cdot, 0, 1)$ is given by

- $|a| = \lceil \log a \rceil$
- $|(a, b)|_+ = \max\{|a|, |b|\}$
- $|(a, b)|_\cdot = |a| + |b|$

This measure is additively bounded.

Proposition (Grädel, D.). Let $\pi : \mathrm{Lit}_{\mathfrak{A}}(\tau) \to K$ be a semiring interpretation with $m = |\max \pi|$ and $n = |A|$. For any first-order formula ψ of depth $d = d(\psi)$ and additively bounded cost measure on K, we have that

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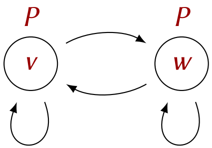
$$|\pi(\psi)| \leq m \cdot n^d.$$

In the case that the cost function of K is multiplicatively bounded, we instead have that

$$|\pi(\psi)| \leq m^{n^d}.$$

These size bounds can actually be realised, even in ML:
Consider the formula $\Box^k P$ on a clique with n nodes, each in P

$\mathcal{K}$:



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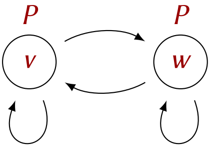

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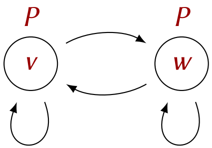

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There are however K-interpretations such that the provenance values of GF formulae are a lot larger than the maximal provenance value of any ML formula. The same goes for the comparison between GF and FO.

Outlook: Guarded Negation Logic

Syntax:

$$\varphi ::= R(\bar{x}) \mid x = y \mid \exists x \varphi \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \alpha(\bar{y}) \wedge \neg\varphi(\bar{y})$$

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Problem: No universal quantification means no negation normal form.

Solution: Close the logic under negation, retaining its nice properties.

$$\begin{aligned}
\varphi^+ ::= & R(\bar{x}) \mid x = y \mid \exists x \varphi^+ \mid \varphi^+ \vee \varphi^+ \mid \varphi^+ \wedge \varphi^+ \mid \\
& \alpha(\bar{y}) \wedge \varphi^-(\bar{y}) \mid \neg\varphi^- \\
\varphi^- ::= & \neg R(\bar{x}) \mid x \neq y \mid \forall x \varphi^- \mid \varphi^- \wedge \varphi^- \mid \varphi^- \vee \varphi^- \mid \\
& \alpha(\bar{y}) \rightarrow \varphi^+(\bar{y}) \mid \neg\varphi^+
\end{aligned}$$

Outlook: Guarded Negation Logic

Let $\pi : \mathrm{Lit}_{\mathfrak{A}} \to K$ be a K-interpretation. We define

$$\pi(\alpha(\bar{a}) \wedge \varphi^{-}(\bar{a})) = \pi(\alpha(\bar{a})) \cdot \pi(\varphi^{-}(\bar{a})))$$

$$\pi(\alpha(\bar{a}) \to \varphi^{+}(\bar{a})) = \begin{cases} 1, & \mathfrak{A} \not\models \alpha(\bar{a}) \\ \pi(\alpha(\bar{a})) \cdot \pi(\varphi^{+}(\bar{a}))), & \text{else} \end{cases}$$

Conclusion

- Provenance analysis can be applied to guarded logics
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Conclusion

- Provenance analysis can be applied to guarded logics
- Provenance analysis for the model checking games for guarded logics is nicely compatible with formula provenance
- For many cost models the complexity of provanence analysis for guarded logics is exponential or even doubly exponential
- However there are still a lot of applications where provenance analysis for the guarded logics is much less complex than for FO