Exploring Sparse Graphs with Advice

H.-J. Böckenhauer[1] **J. Fuchs**[2] W. Unger[2]

[1]Department of Computer Science
ETH Zürich

[2]Computer Science 1
RWTH Aachen University

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Outline

Online Graph Exploration

Task: find a shortest tour visiting every vertex of an unknown graph G at least once.

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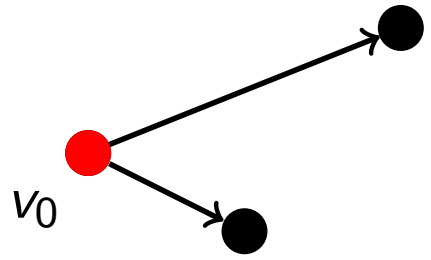

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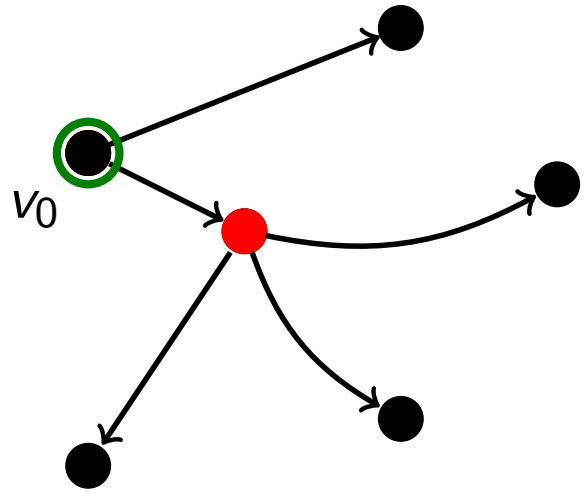

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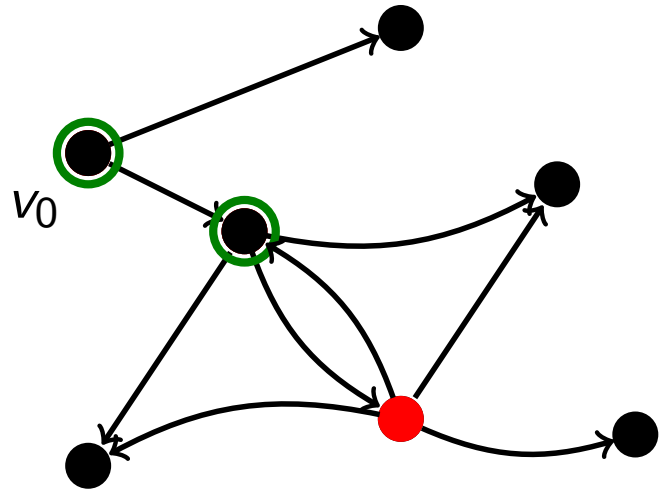

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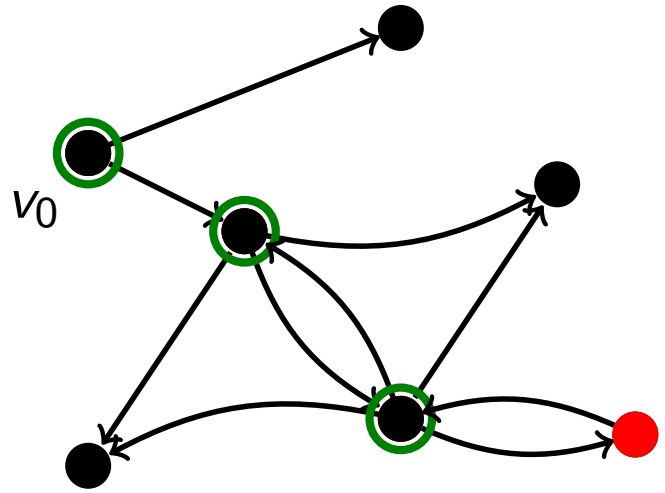

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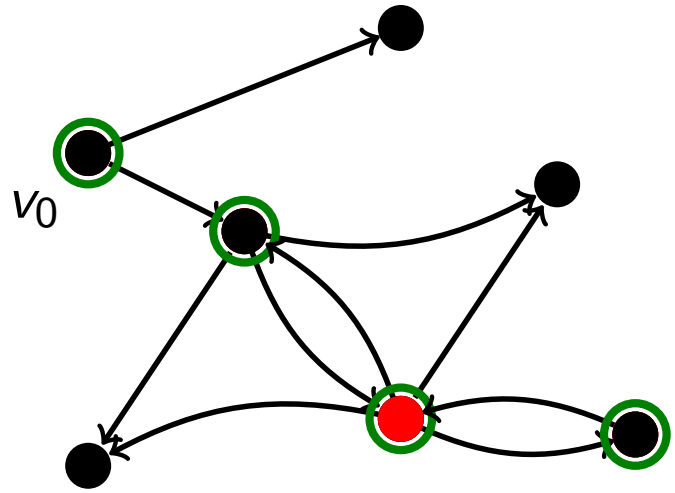

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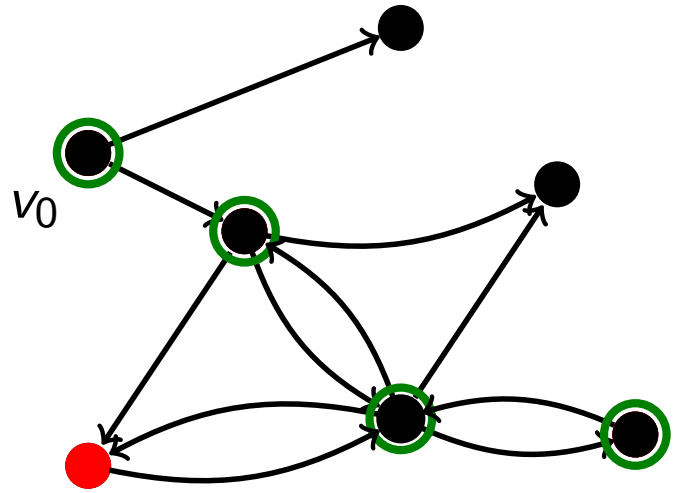

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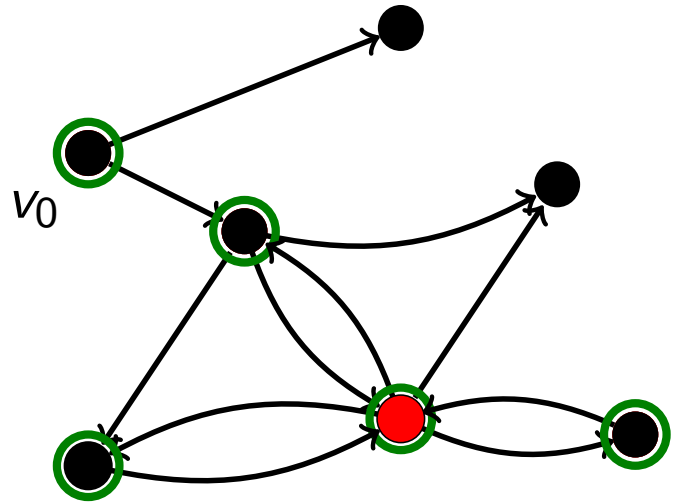

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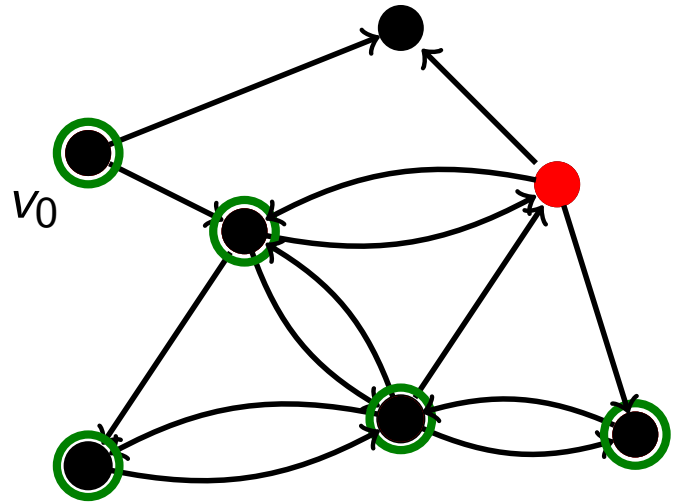

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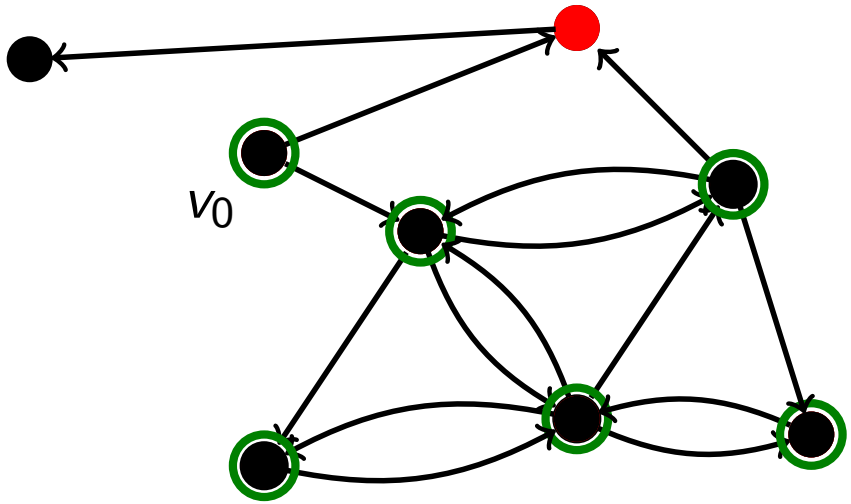

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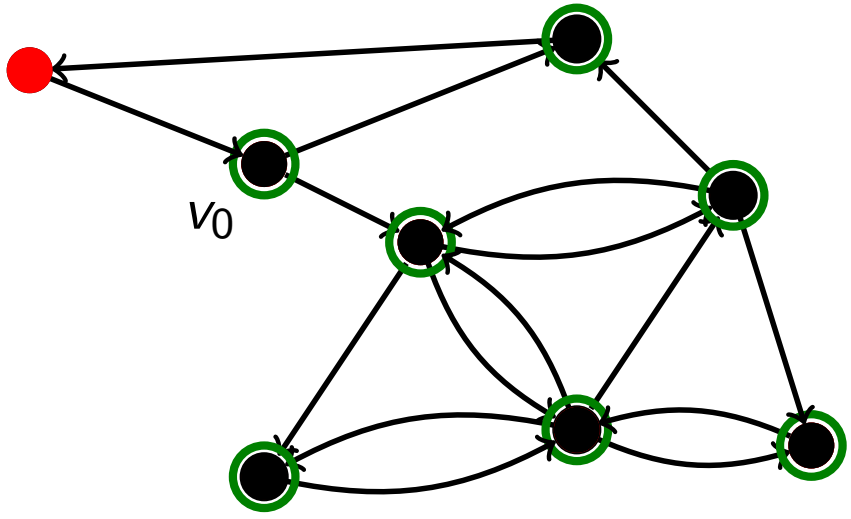

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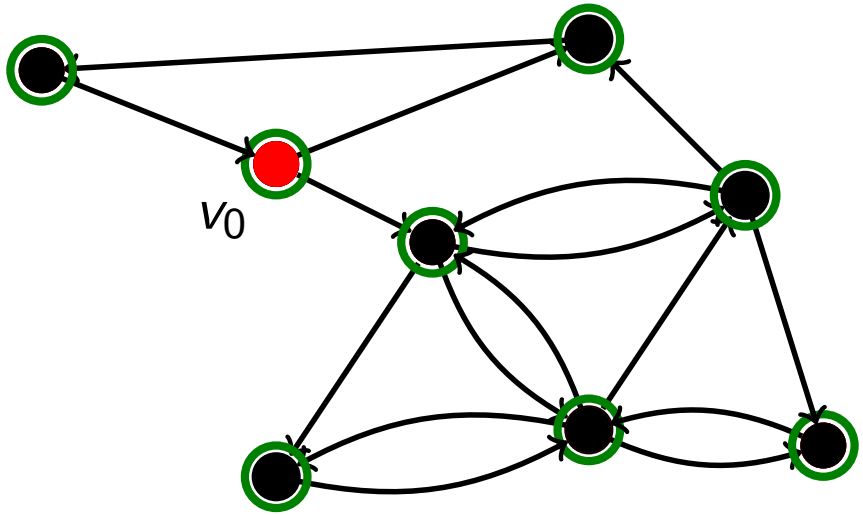

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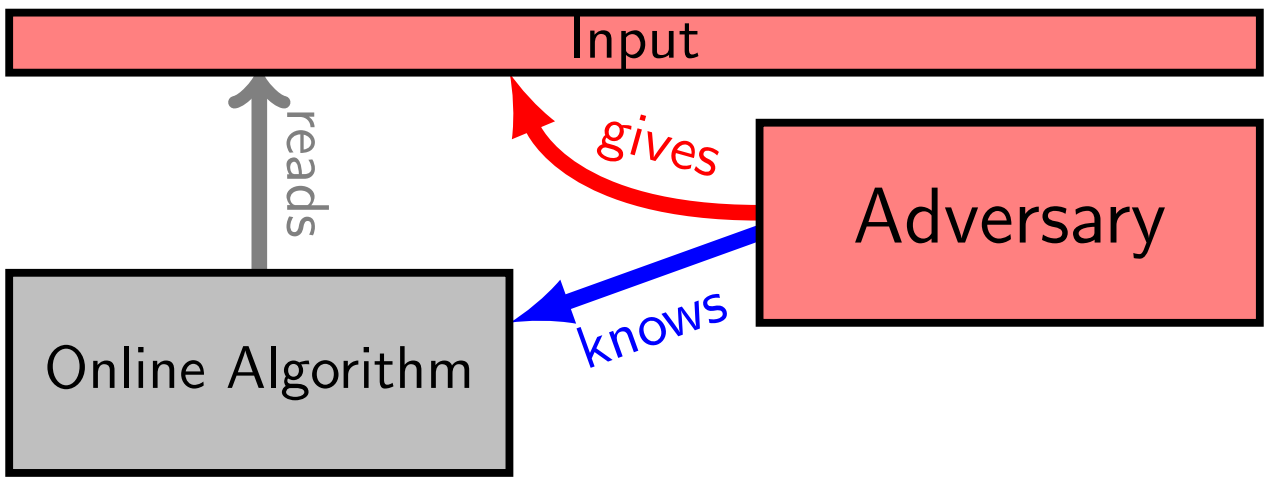
Algorithm can "buy" information about the future.

The Oracle

Algorithm and oracle are a team and play against an adversary.

The classical setting:

- The algorithm receives the input step by step.
- Adversary knows algorithm and prepares worst-case input.



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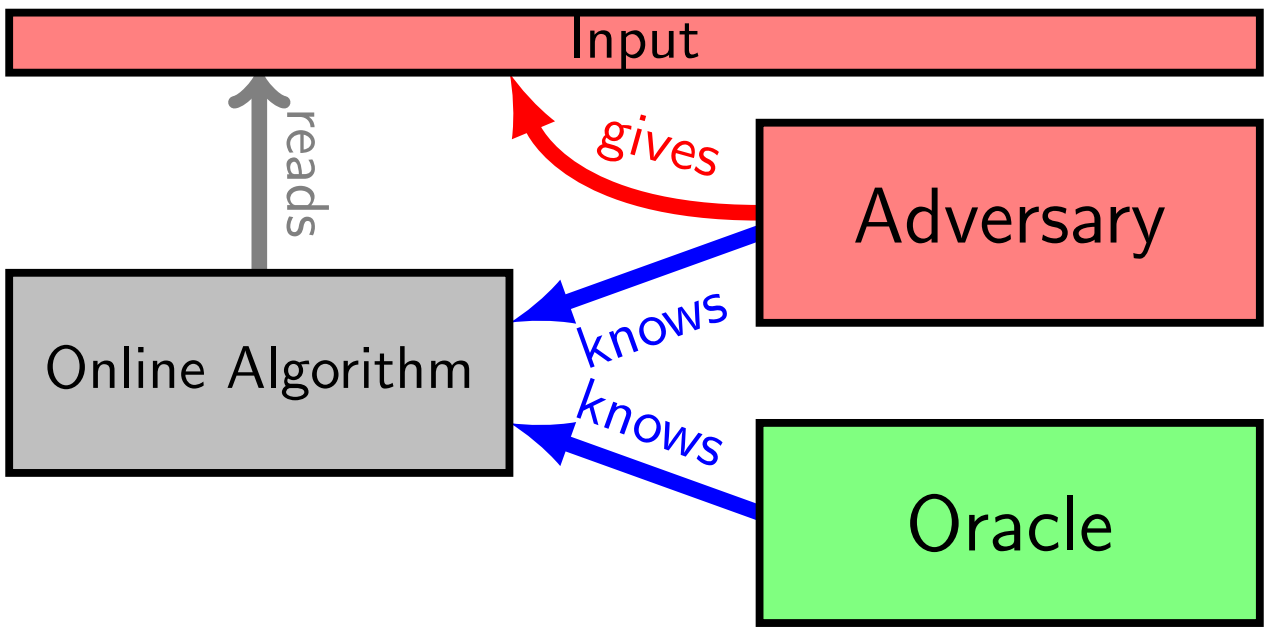

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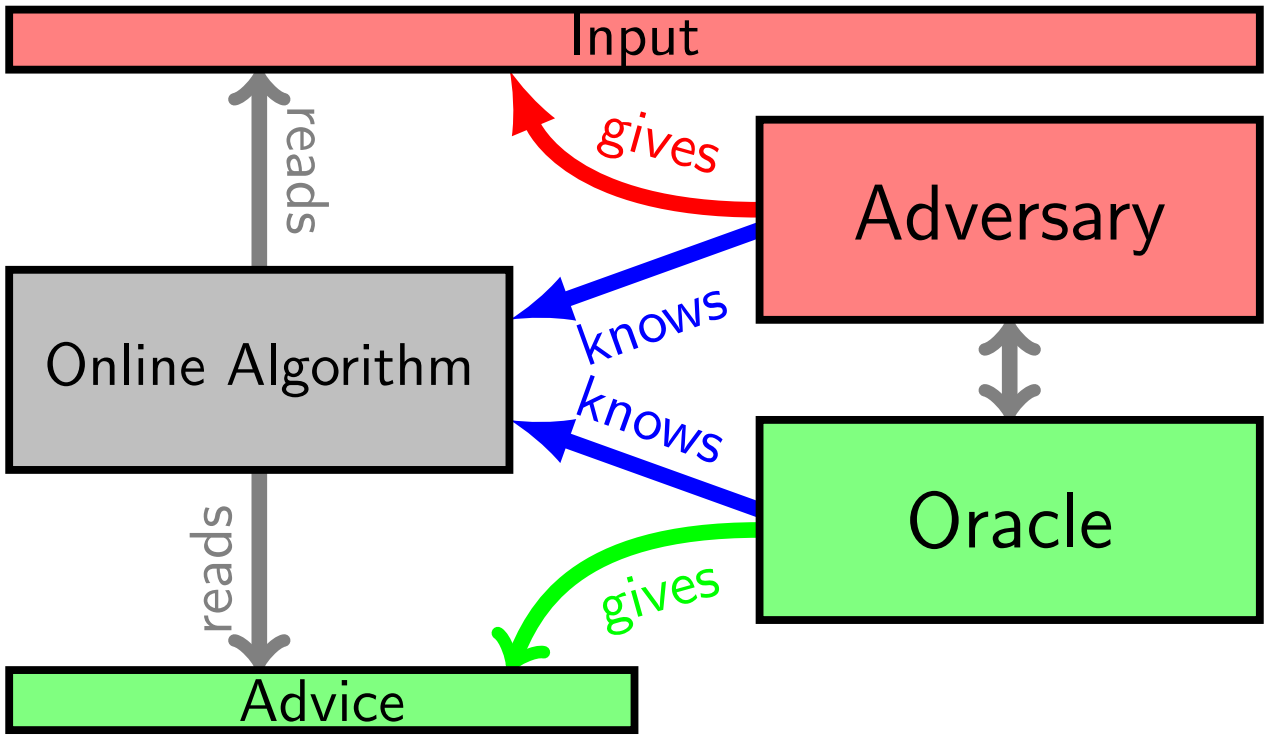

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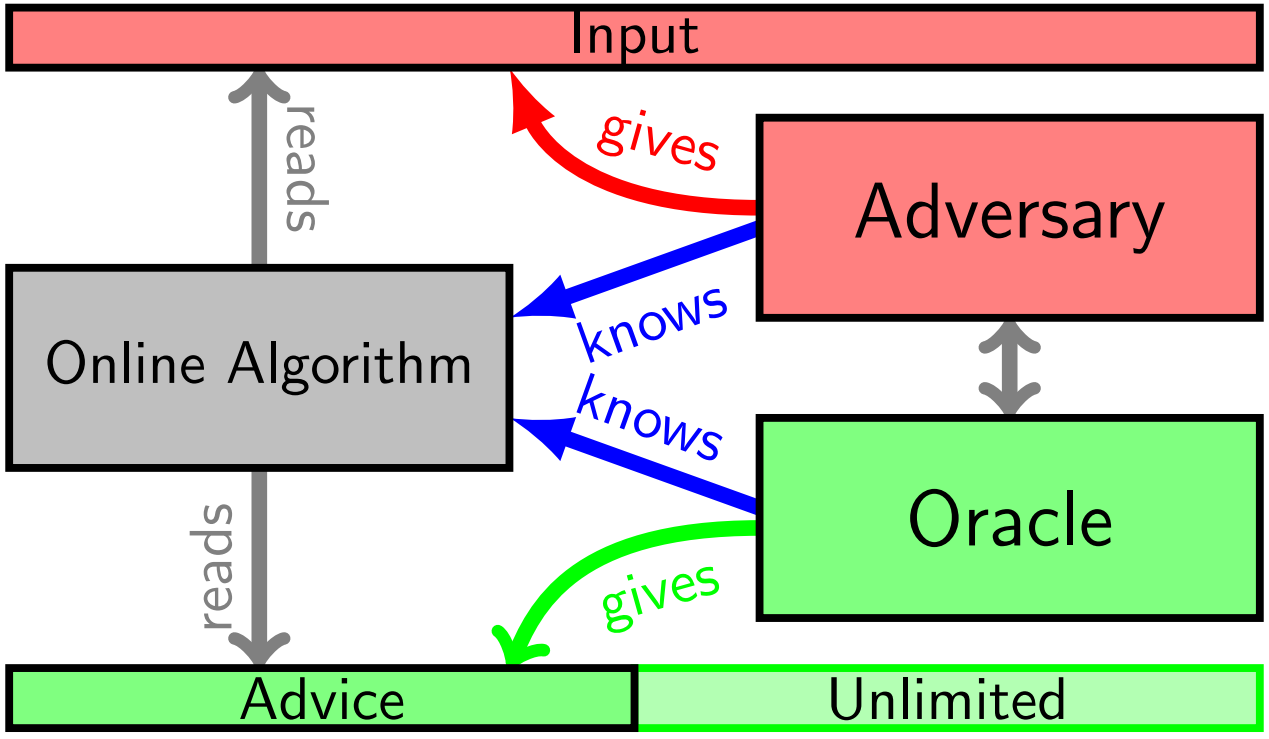

Problem and Setting Together

- Adversary constructs G.
- Oracle knows G and the optimal exploration tour S^*.
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We will show an algorithm that needs at most $\mathcal{O}(m)$ bits of advice.

Edges

For sake of simplicity, we assume that every vertex in G has at most two incoming edges and at most two outgoing edges.
$G = (V, E)$ is strongly connected with $|V| = n$ and $|E| = m$.

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The edges are classified with respect to S^*:

- E_0: Set of edges that are never used in S^*.
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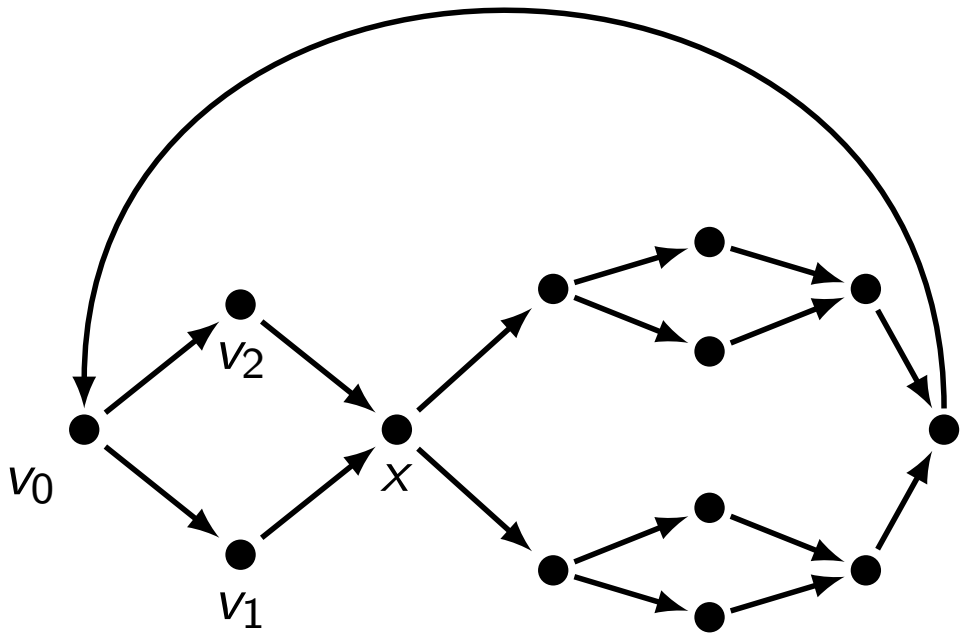
$\rightarrow \log(3) \cdot m$ bits of advice.

The Optimal Solution S^*

Is S^* or the classification into the sets E_0, E_1, E_{Multi} unique?



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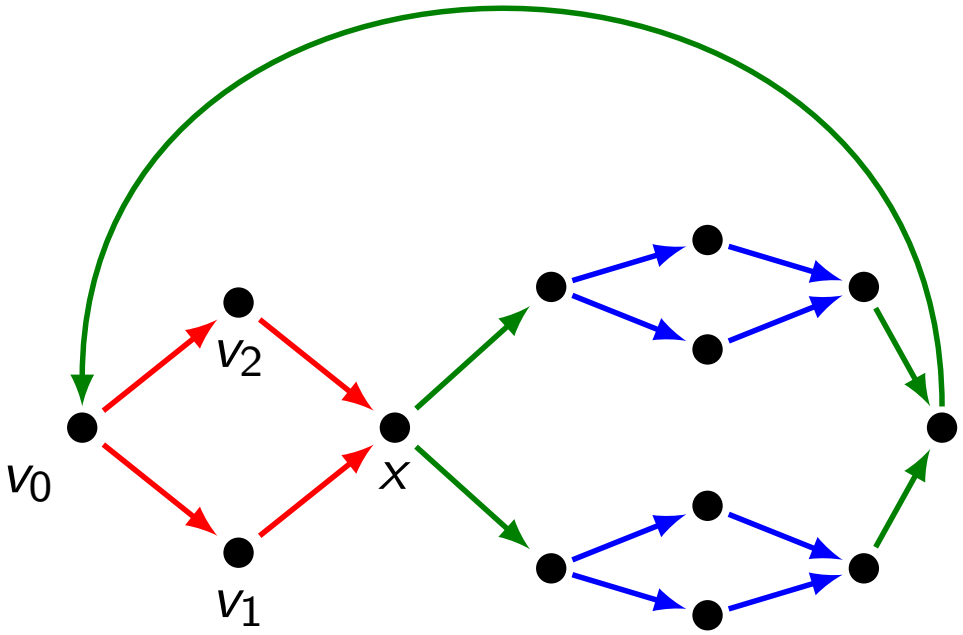

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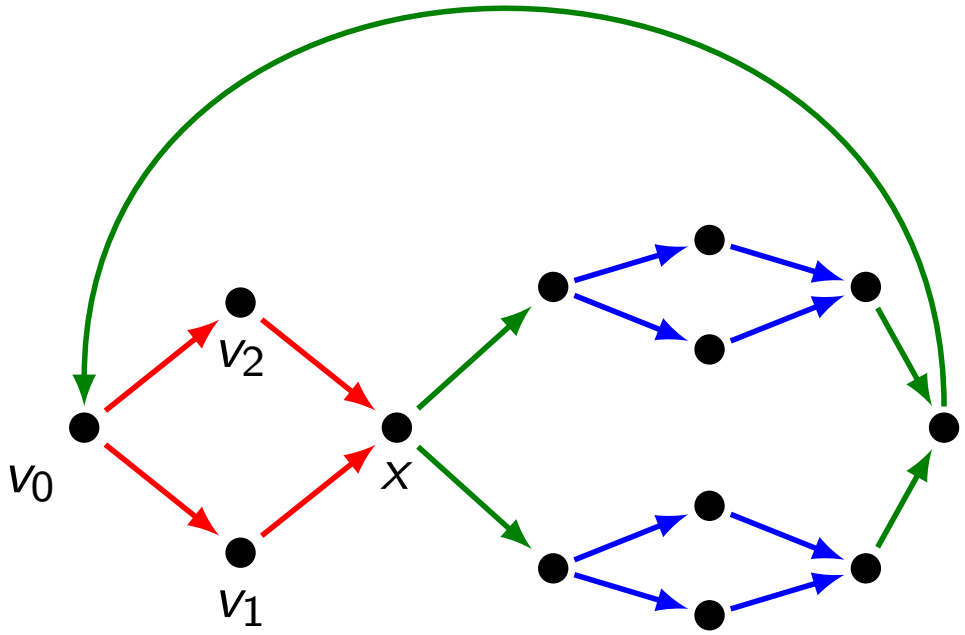
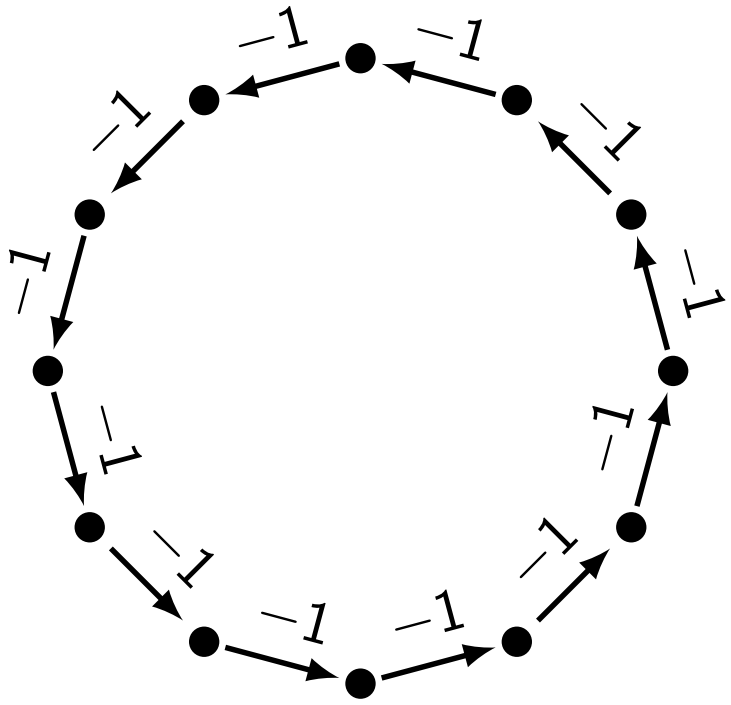


But the oracle can choose an optimal solution S^* that fulfills certain properties.

Properties of S^*

The oracle fixes an optimal solution such that the edges from E_{Multi} do not form a cycle.

Obviously the edges from E_{Multi} cannot form a directed cycle.



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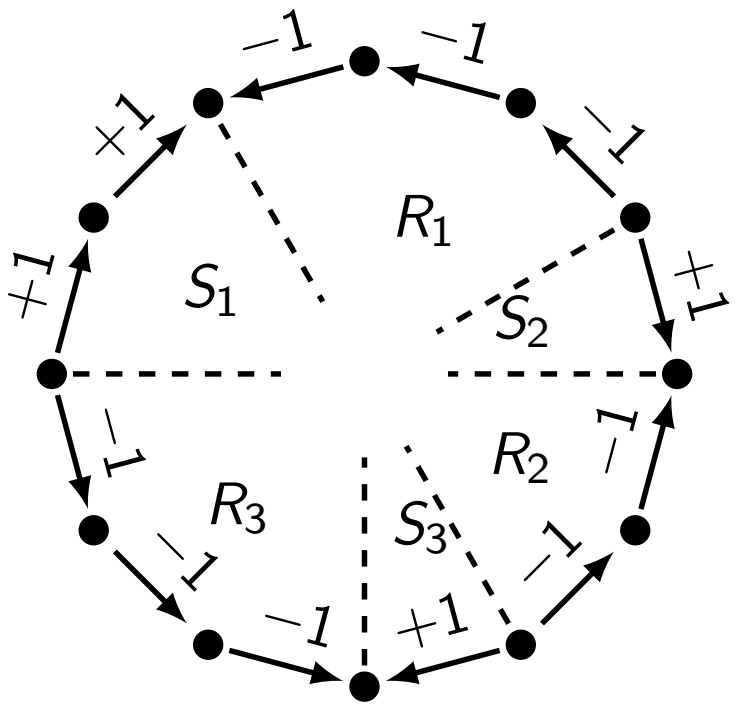

R_i and S_i are paths of edges from E_{Multi} to the same vertex.

The Graph Structure

The graph $F = (V, E_{Multi})$ is a forest.
Even the underlying undirected graph of F is cycle-free.

The Graph $G_{S^*} = (V, E_1 \cup E_{Multi})$ contains cycles.
Thus, every cycle contains at least one edge from E_1.

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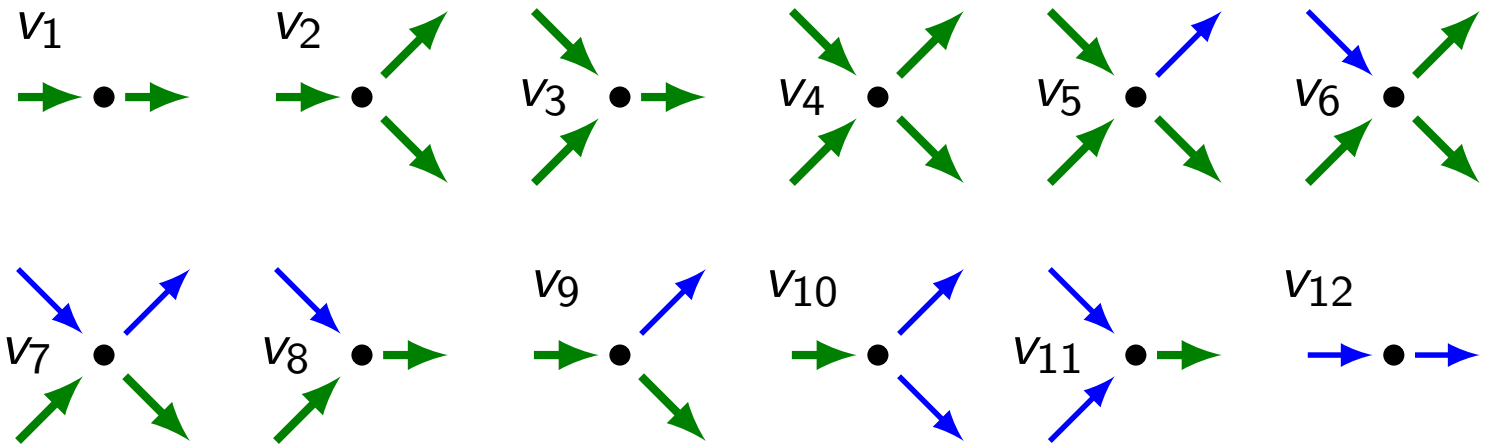
There are only twelve edge configurations possible:



Rough Idea

A vertex that is visited x times in S^* is part of $x - 1$ interchangeable cycles.

Last traversal is part of the "main" cycle that starts and ends at v_0.

The interchangeable cycles can be traversed in any order.
The algorithm needs to be careful:

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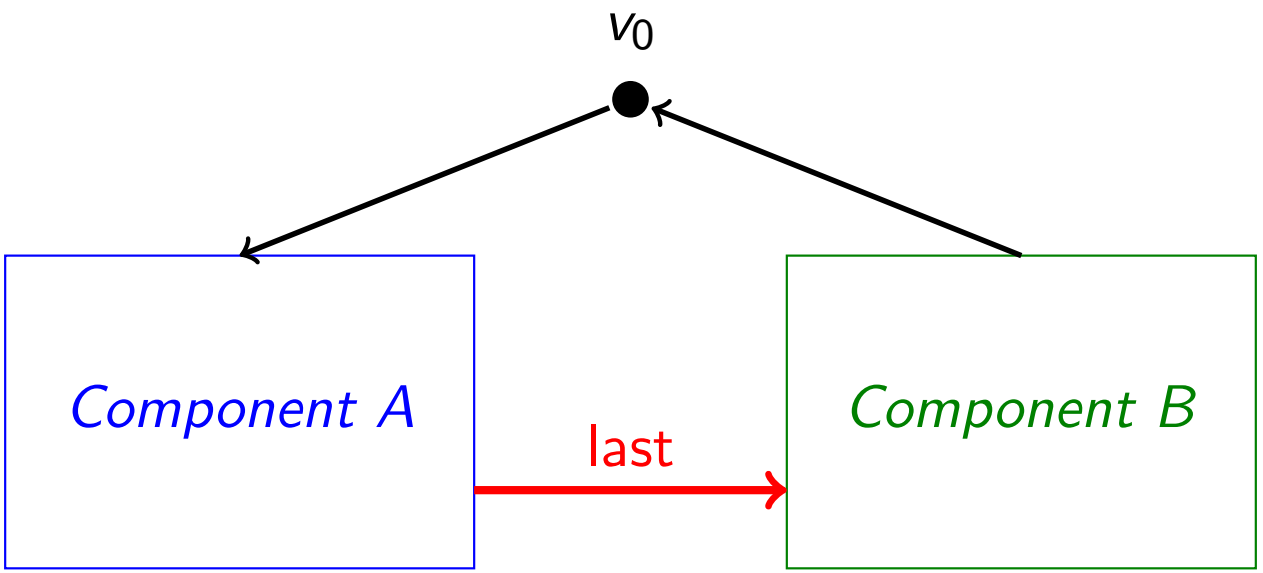
The interchangeable cycles can be traversed in any order.
The algorithm needs to be careful:

- Finish all interchangeable cycles before continuing the "main" cycle.
- Edges cannot be traversed more often than in S^*.

What can go wrong?

The algorithm needs to avoid to cut off components.

Edges that are used after some component is completely explored are called *last*. (These edges continue the "main" cycle)



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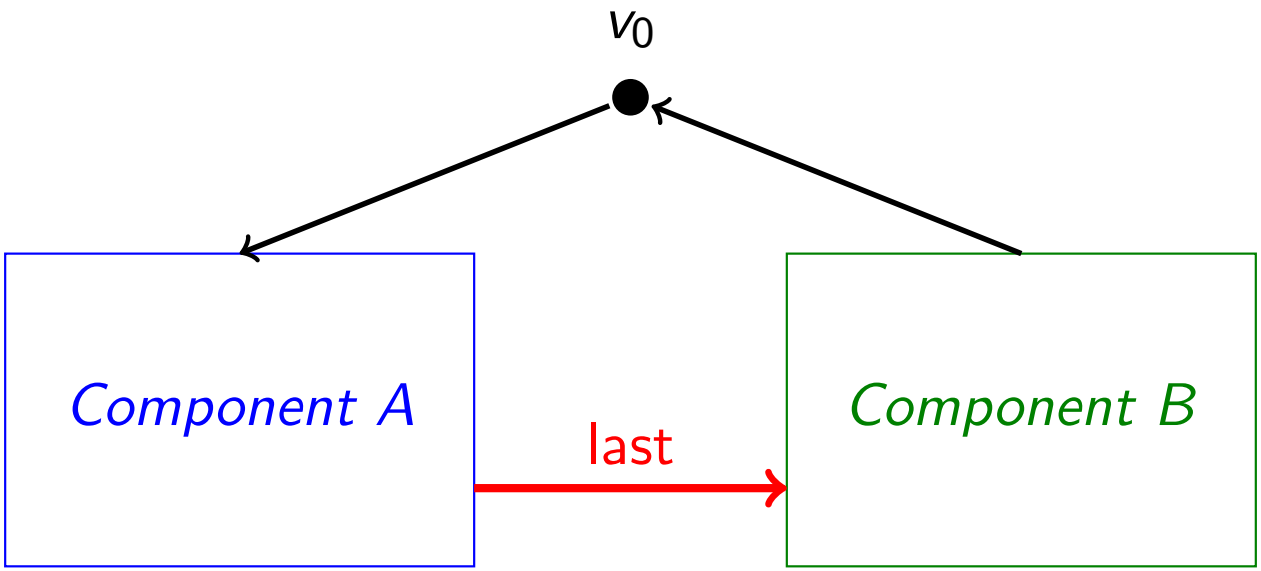

The algorithm asks for every vertex once, which of the two outgoing edges is *last*. $\rightarrow$ n bits of advice.

Number of Traversals $\#_{S^*}$

$2n + \log(3)m$ bits of advice tell the algorithm which edges are used precisely once, never, more than once and which edges are *last*.

Problem: $\#_{S^*}(e)$ for an edge $e \in E_{Multi}$ is unknown.

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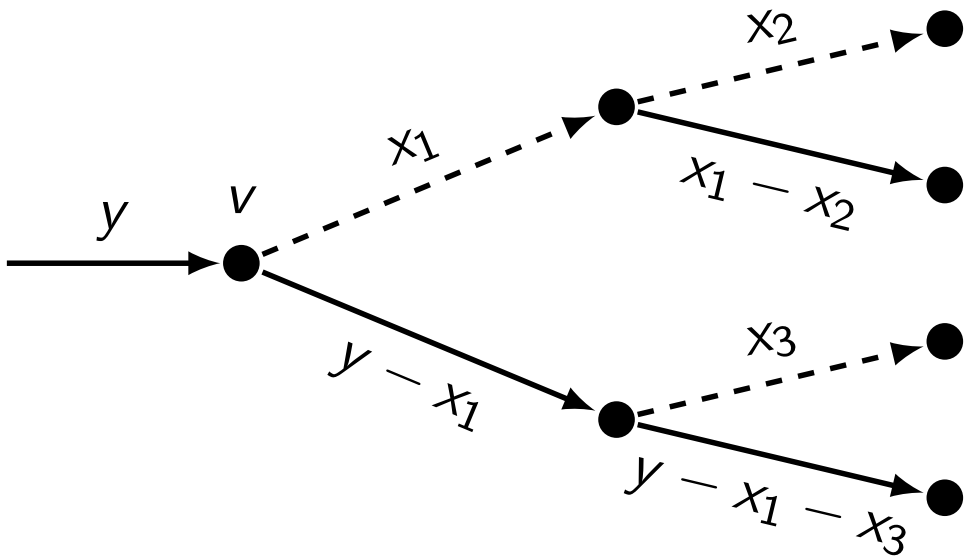
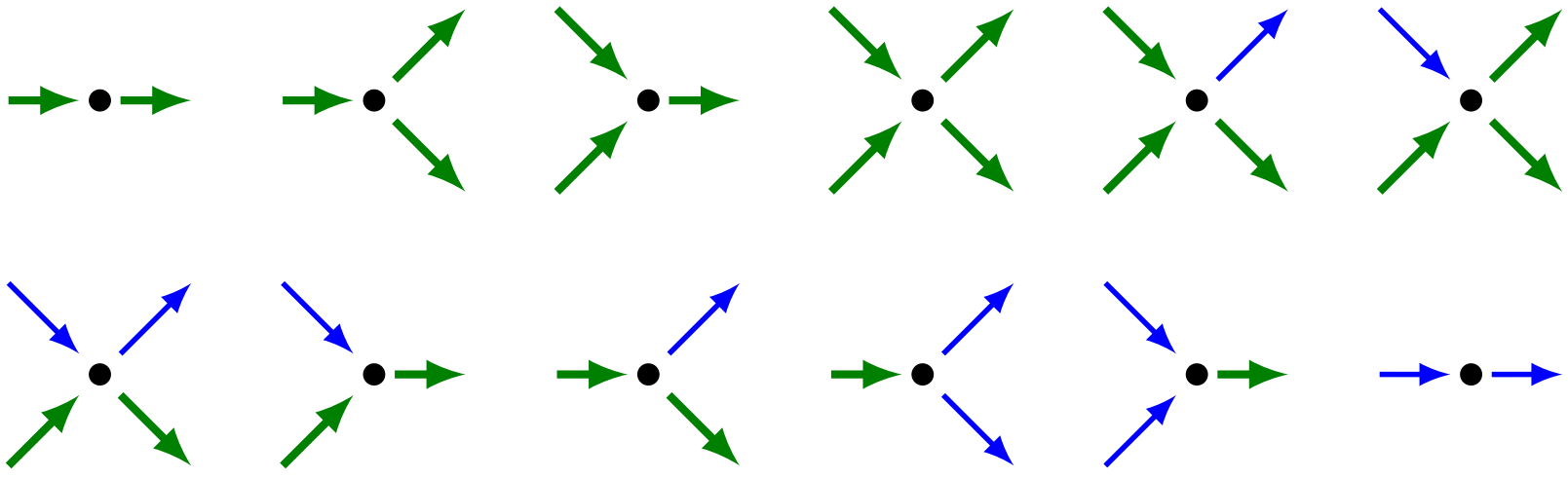
The numbers of traversals influence each other.



If $\#_{S^*}(e)$ is known, the algorithm will not make any mistake.

Heavy and Light Edges

Remember the edge configuration for $G_{S^*} = (V, E_1 \cup E_{Multi})$.



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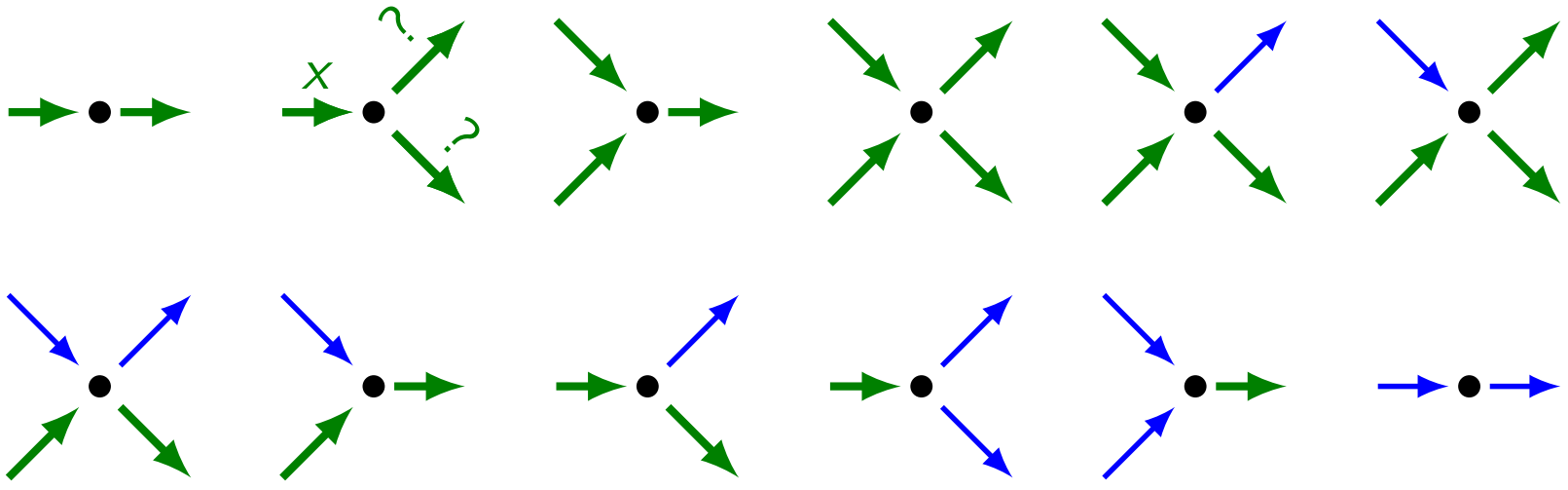

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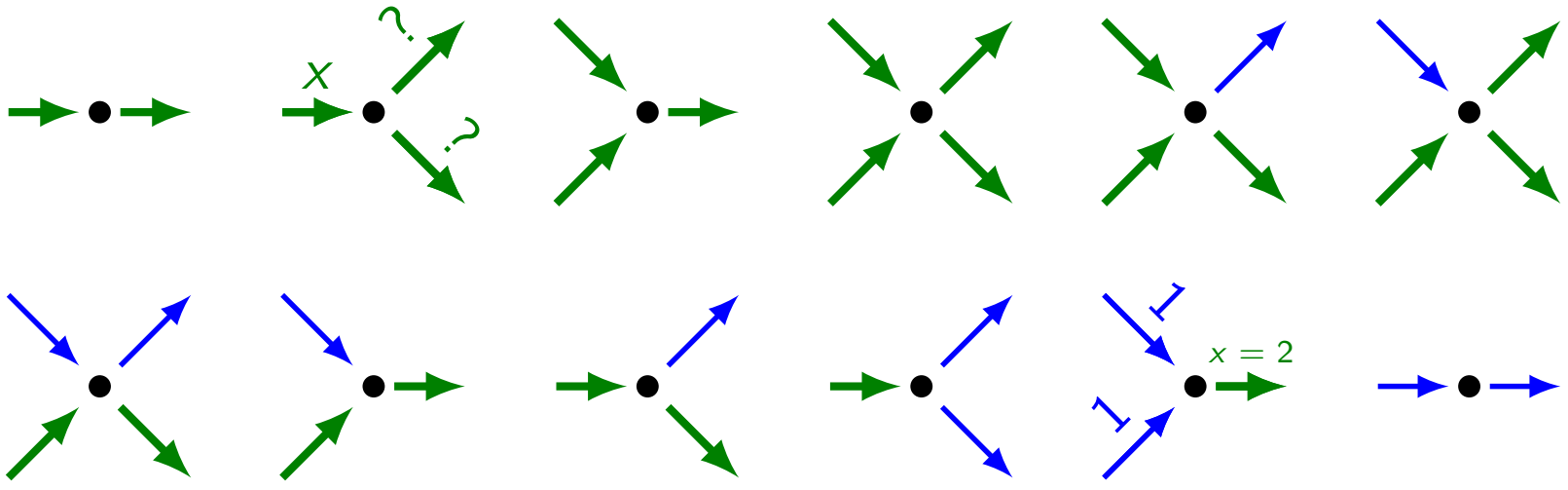

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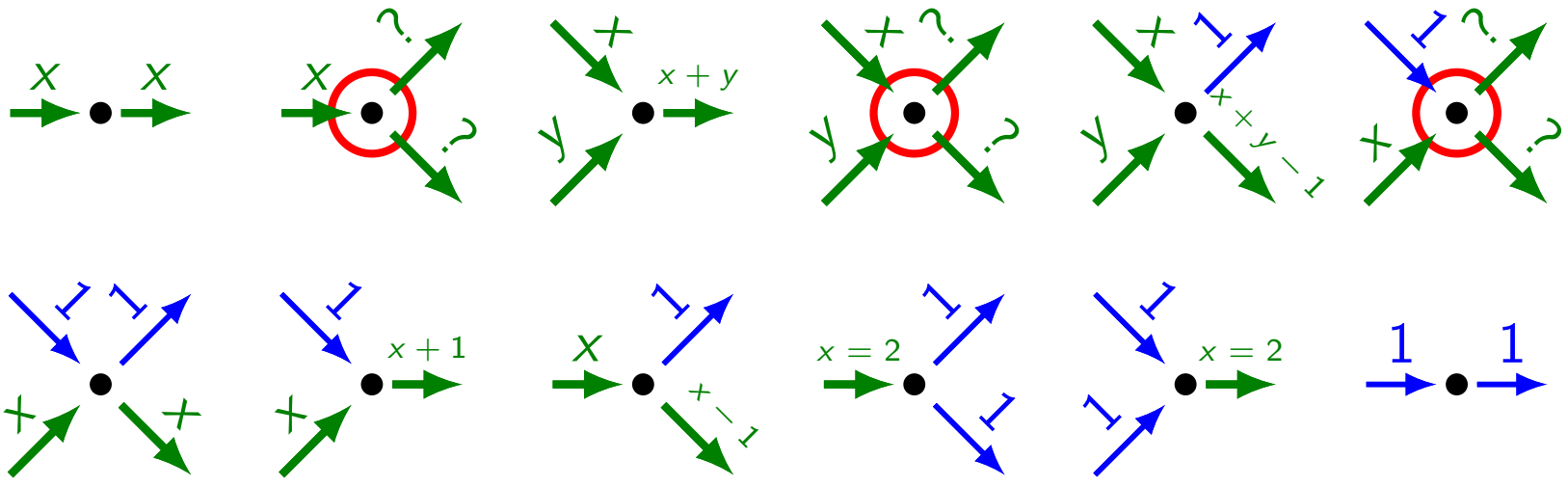

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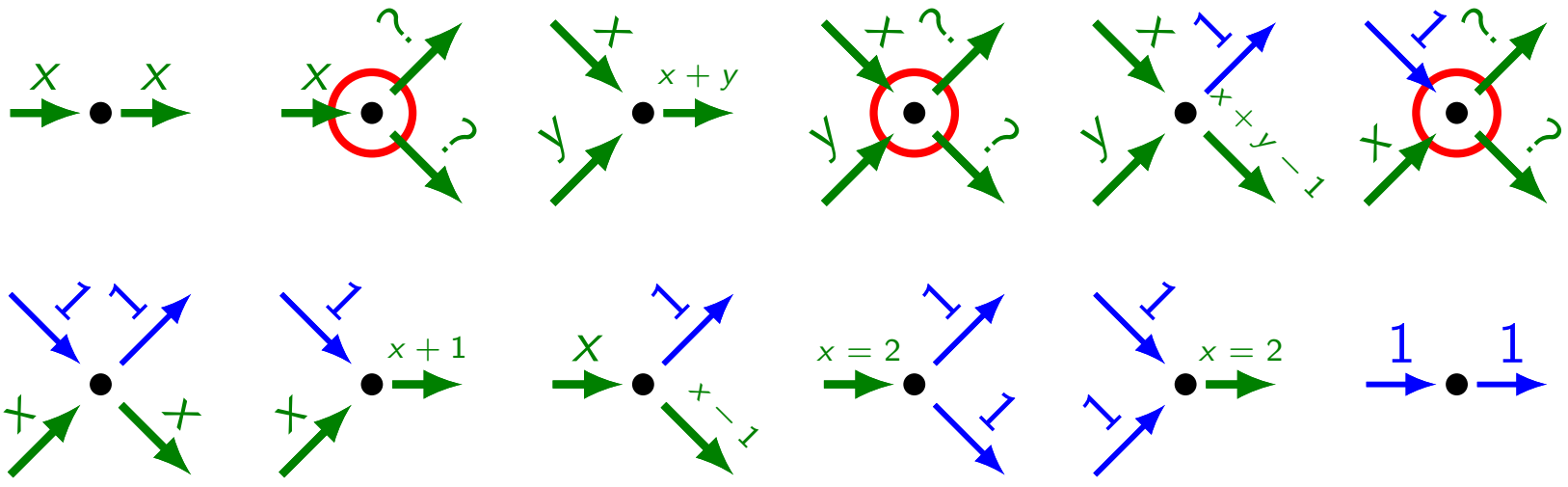
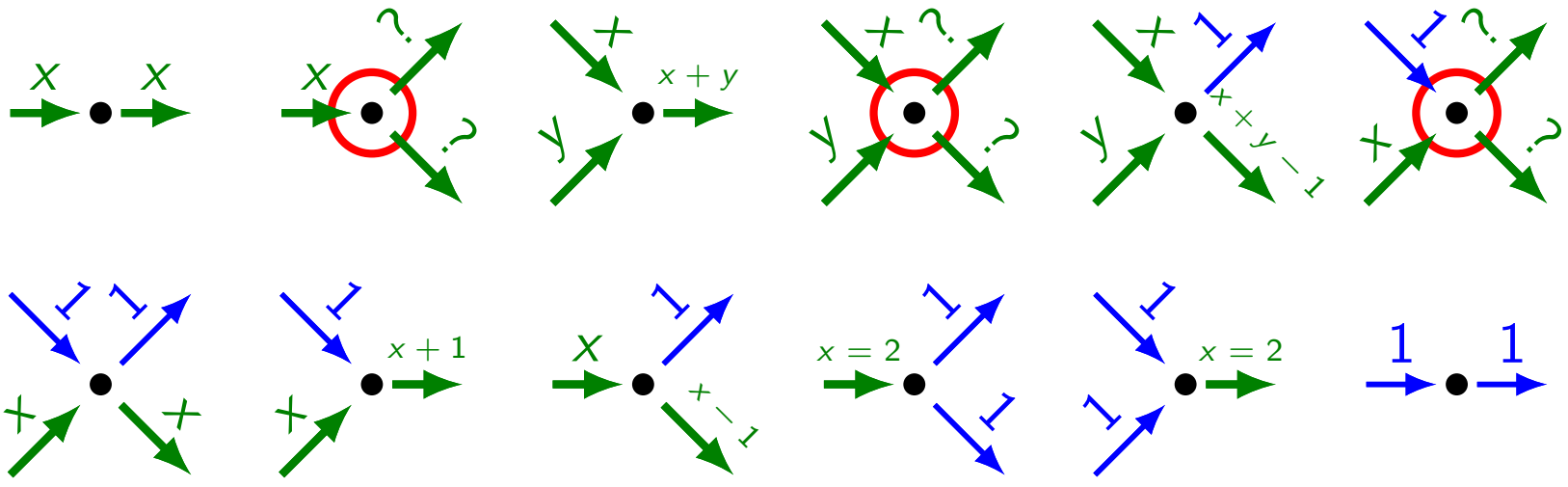


Advice is needed if algo can choose between two edges from E_{Multi}.

Ask which edge is less often used: $\#_{S^*}(e_1) \leq \#_{S^*}(e_2)$?

Heavy and Light Edges

Remember the edge configuration for $G_{S^*} = (V, E_1 \cup E_{Multi})$.



Advice is needed if algo can choose between two edges from E_{Multi}.

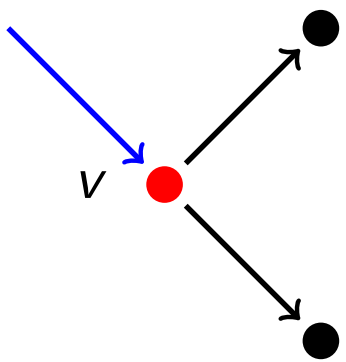
Ask which edge is less often used: $\#_{S^*}(e_1) \leq \#_{S^*}(e_2)$?

The less often used edge is called *light*.
Additionally, the algorithm asks for its precise number of traversals.

Traversing the Edges from E_{Multi}

The algorithm follows the path of *heavy* edges.
(Relatively large and unknown number of traversals)

The algorithm enters a vertex v for the first time.



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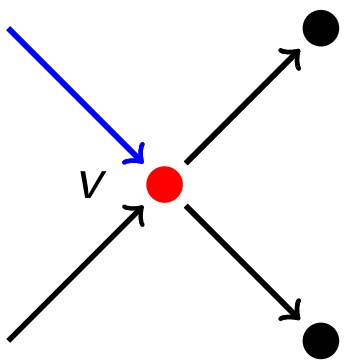

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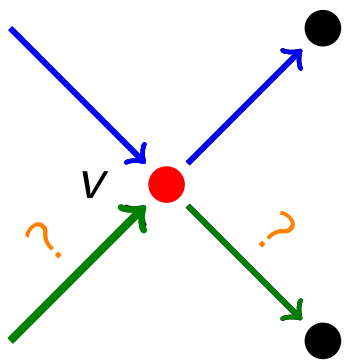

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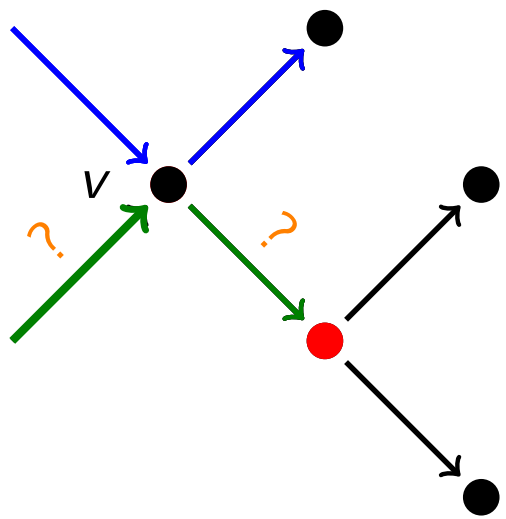

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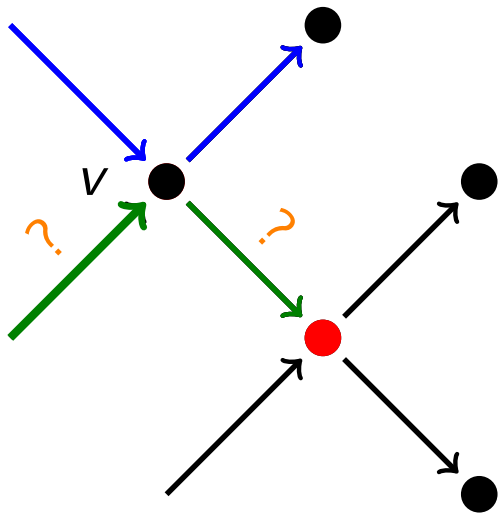

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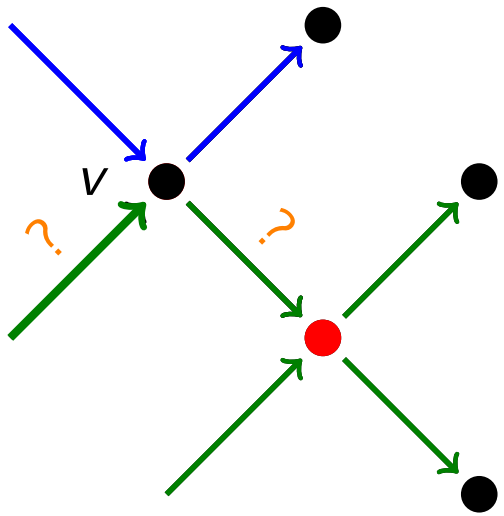

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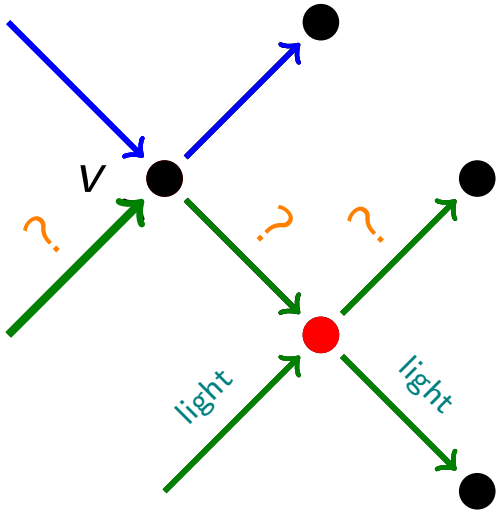

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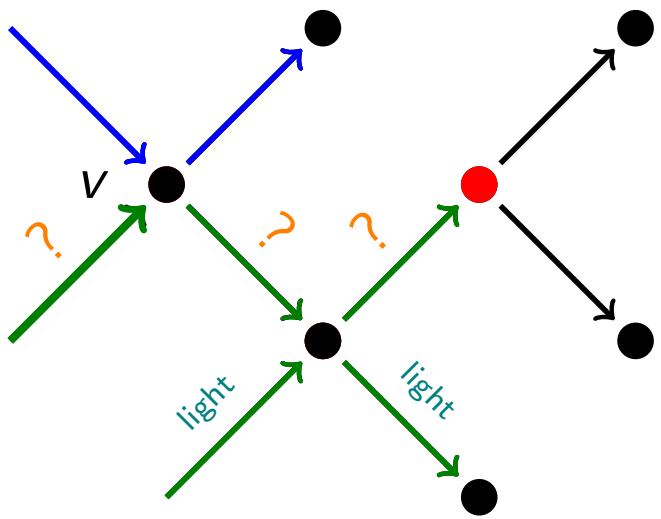

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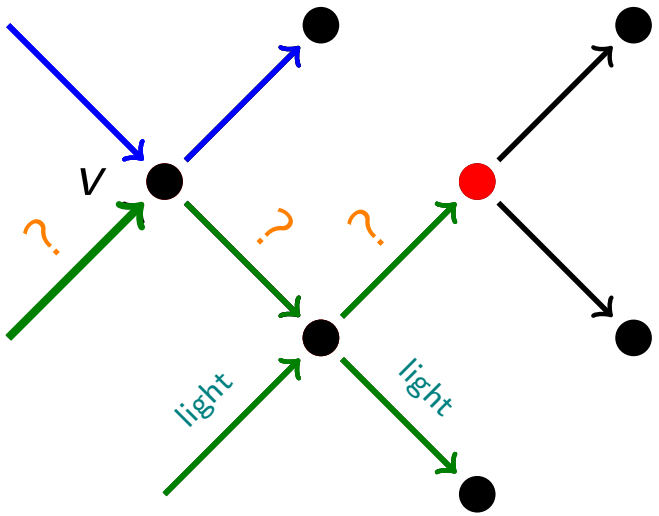

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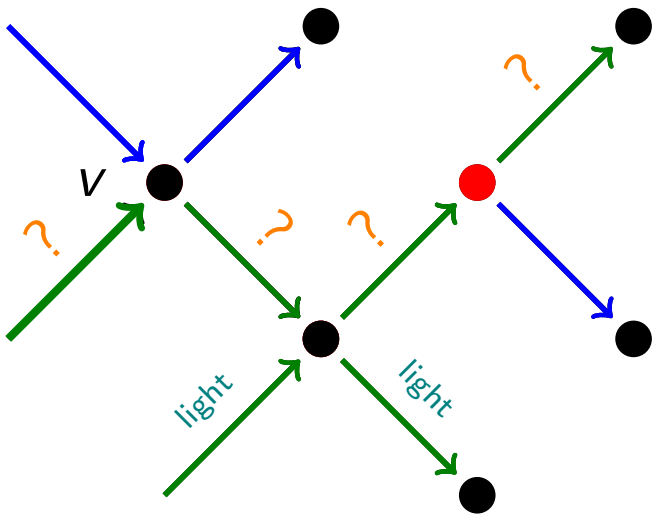

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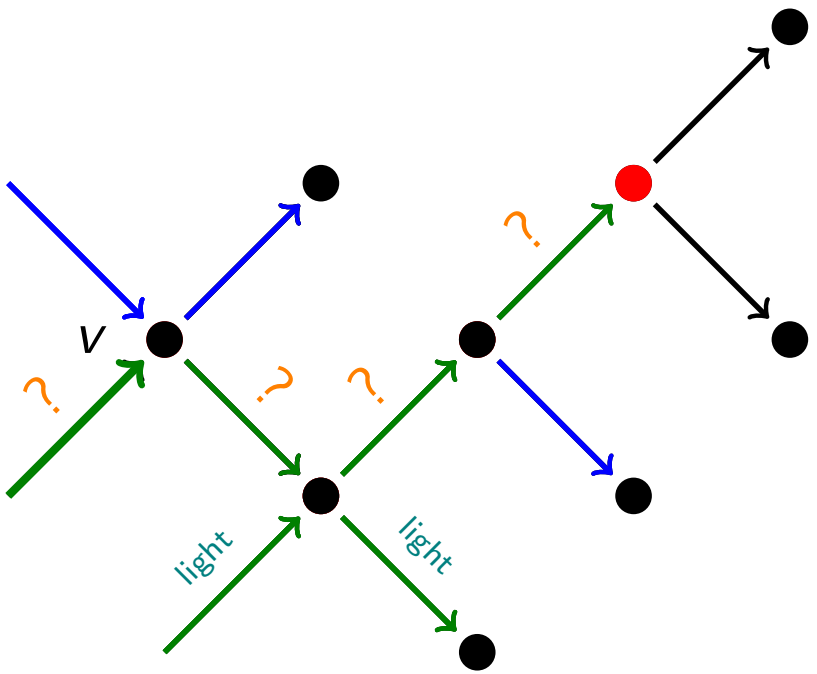

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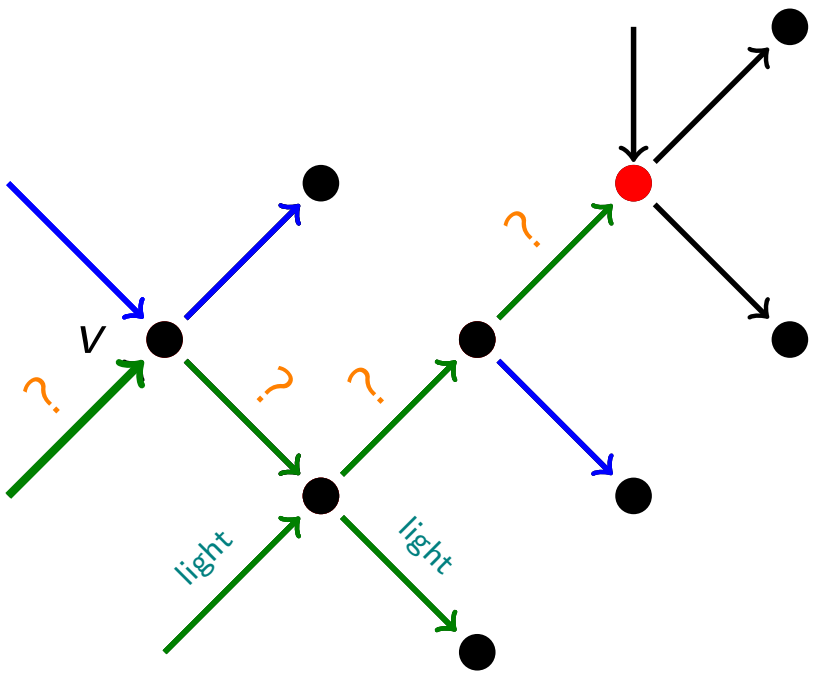

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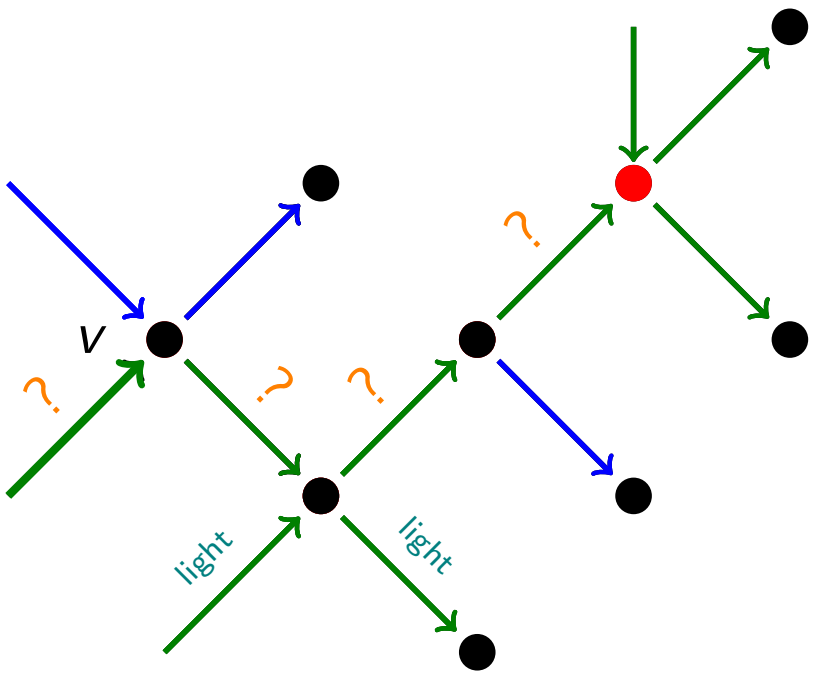

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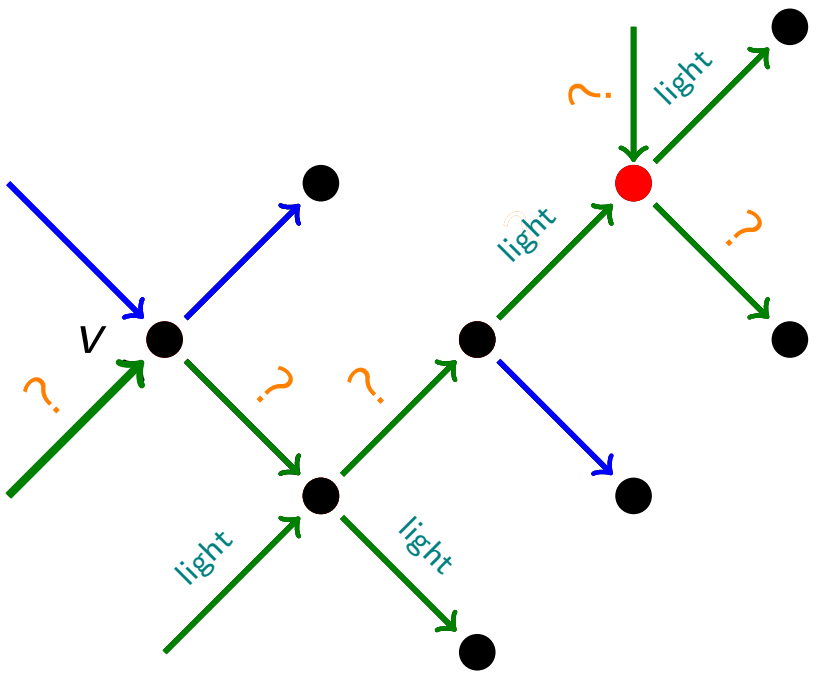

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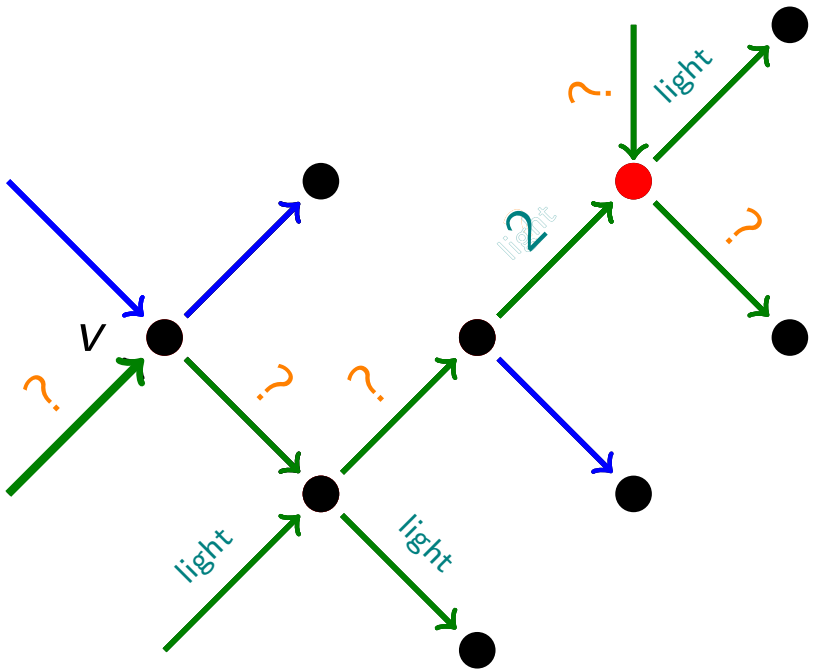

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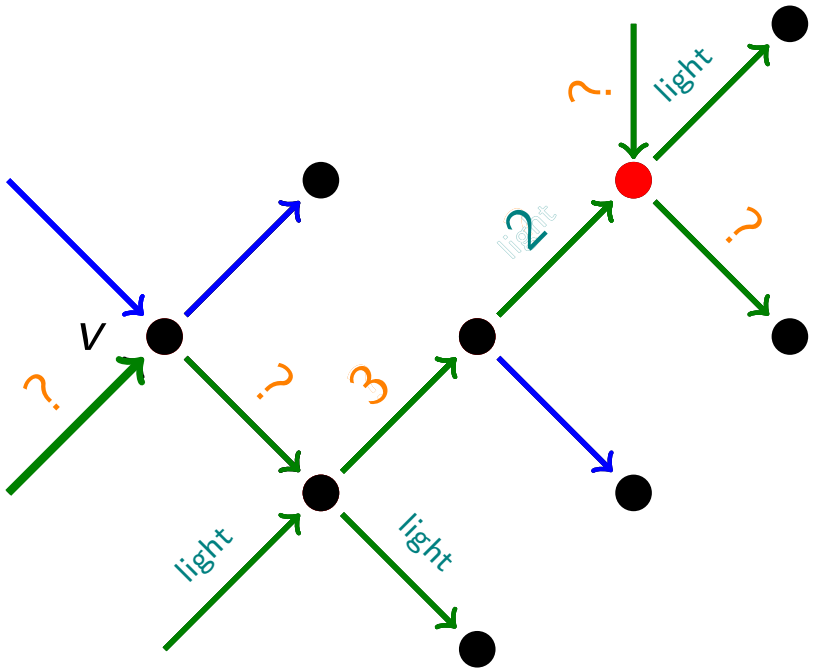

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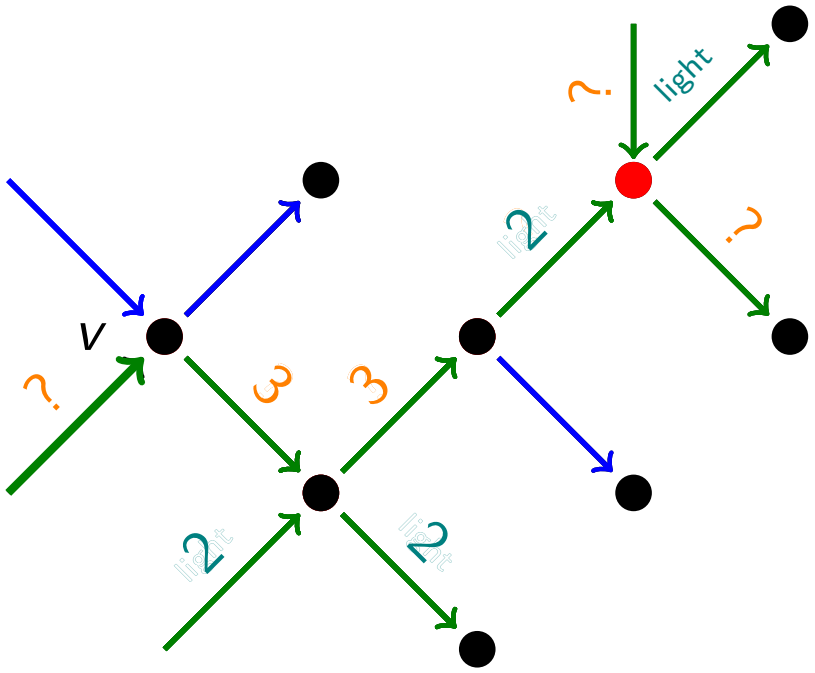

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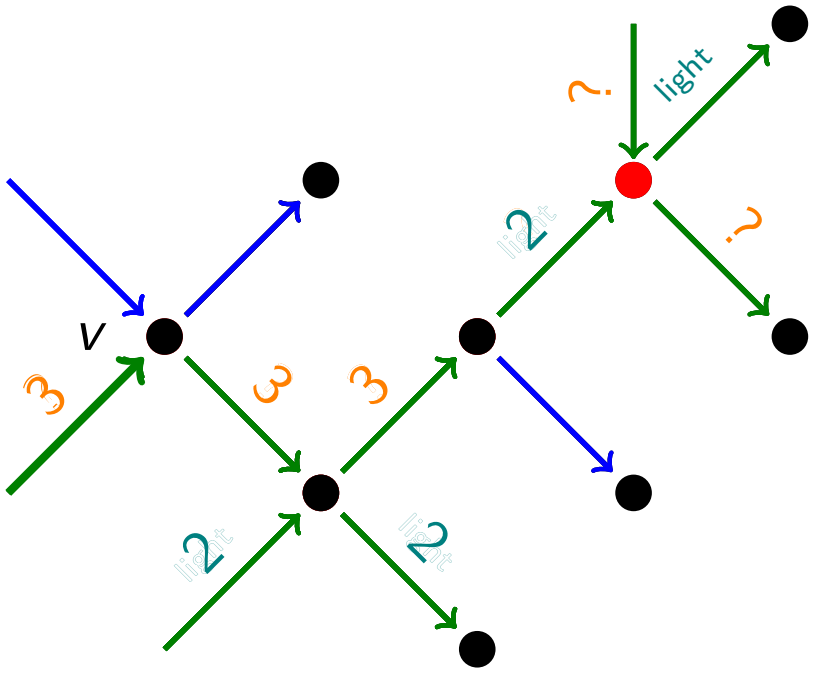

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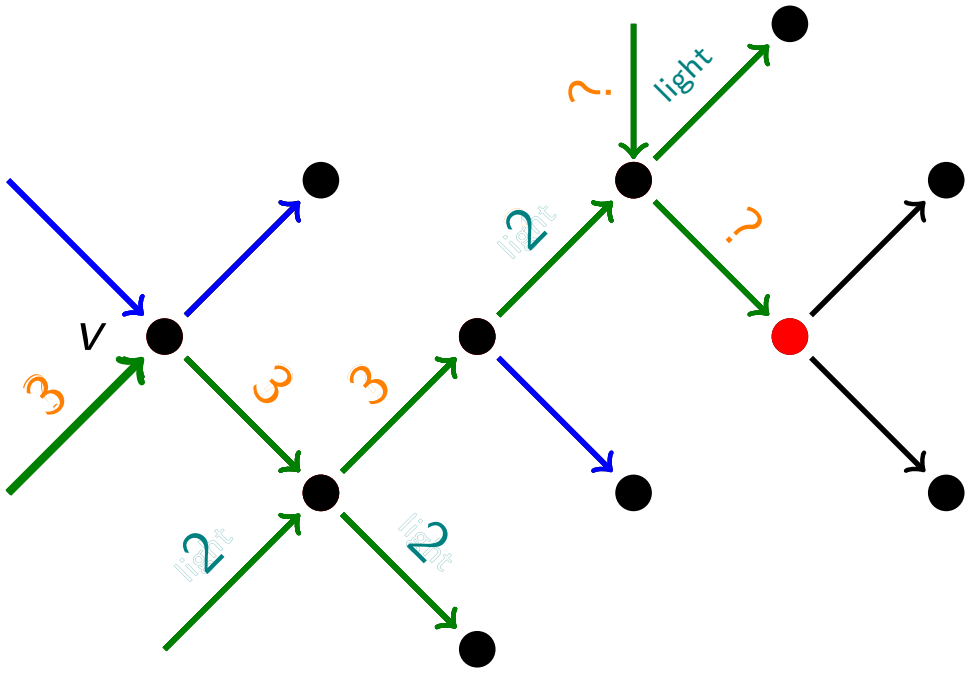

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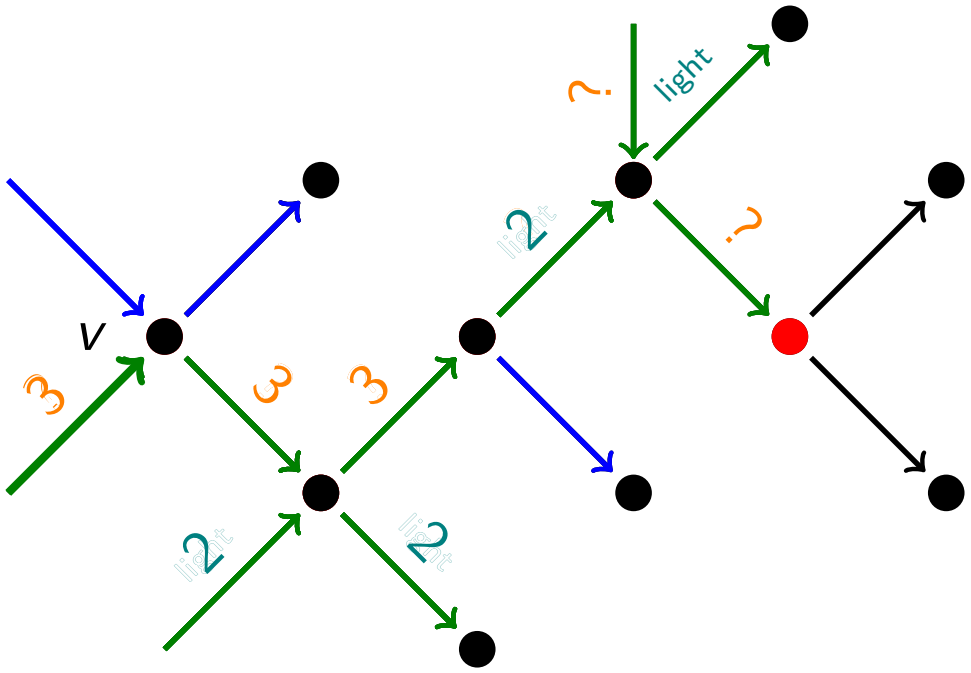

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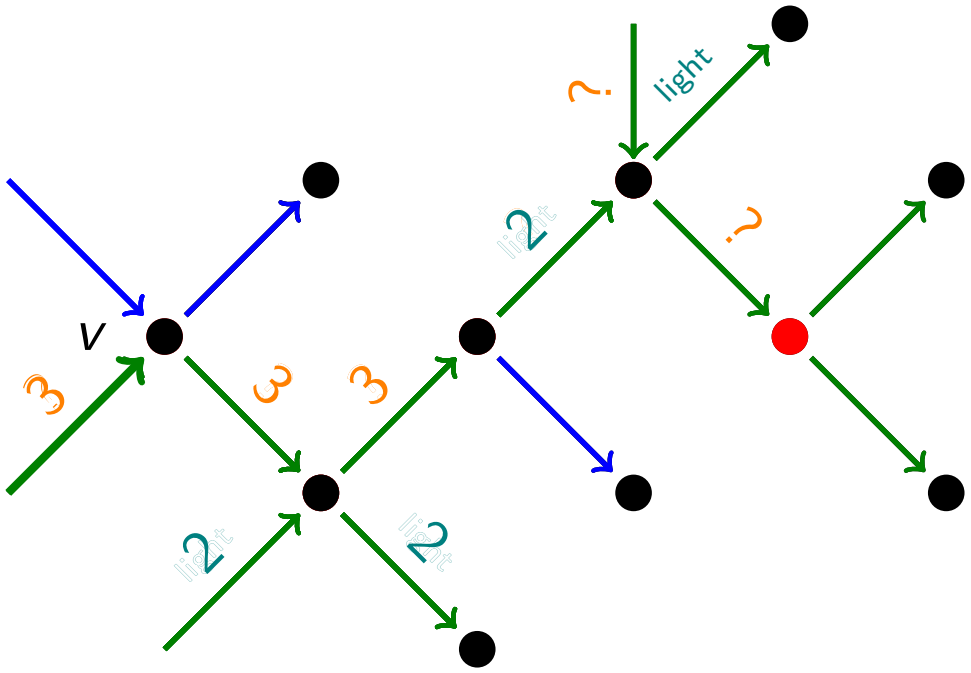

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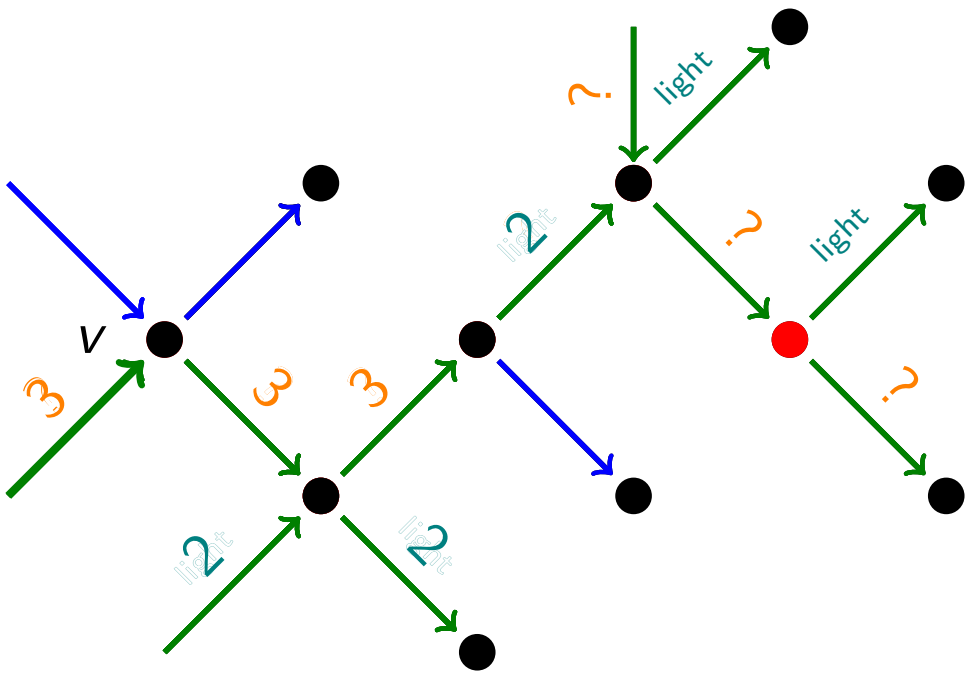

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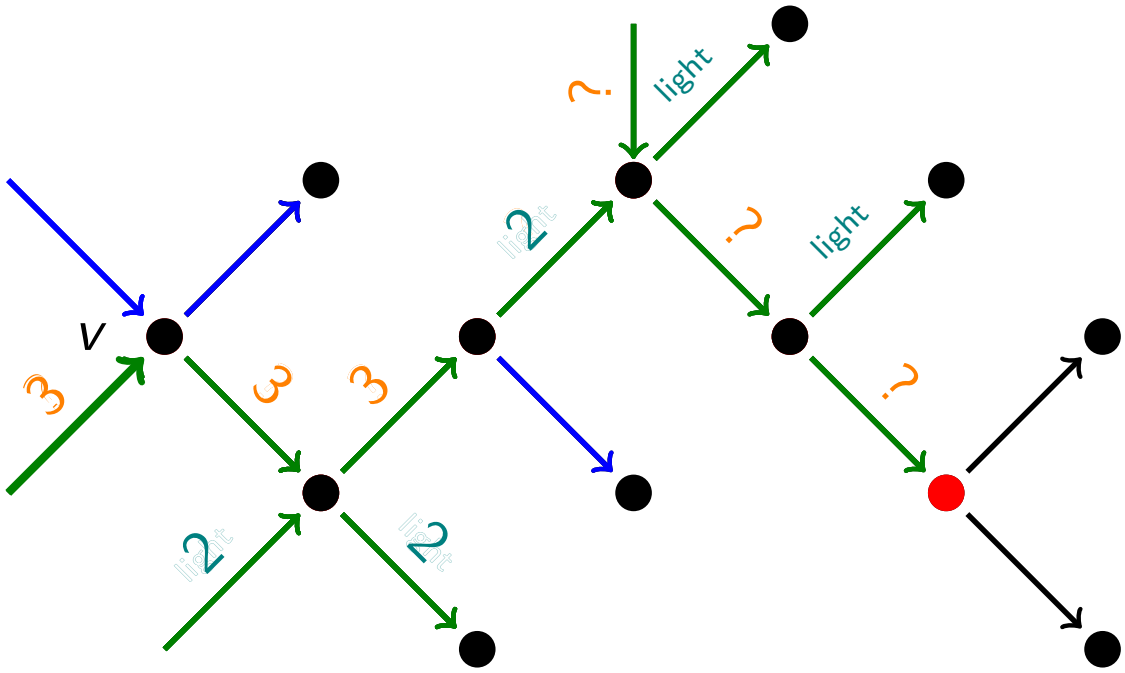

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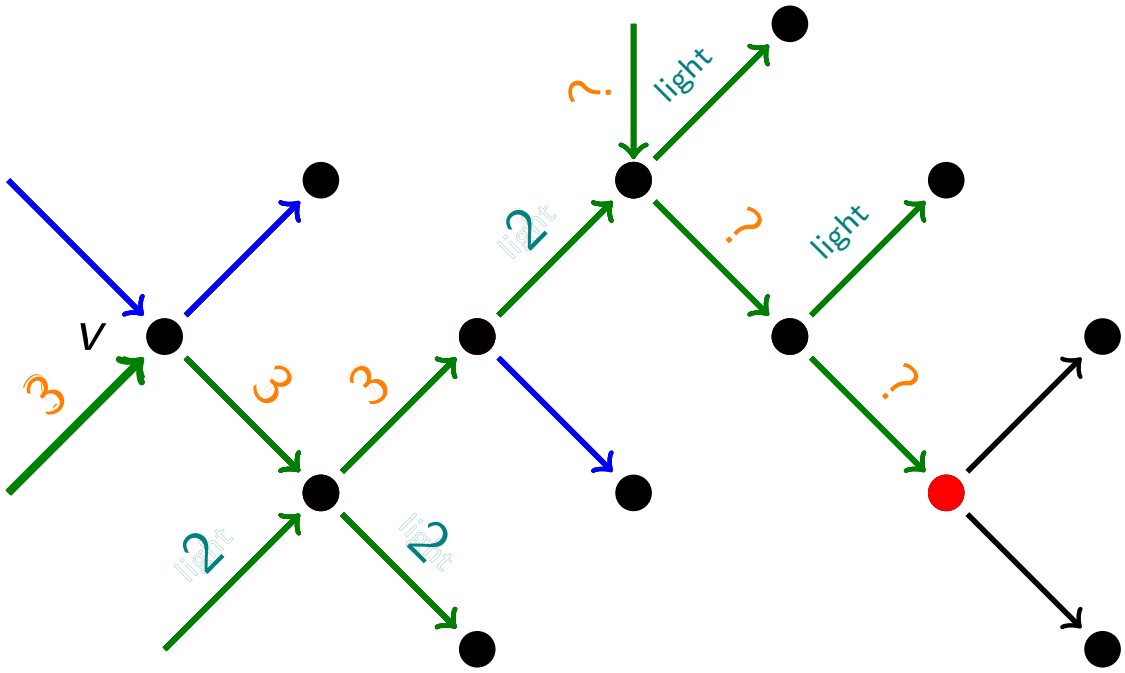

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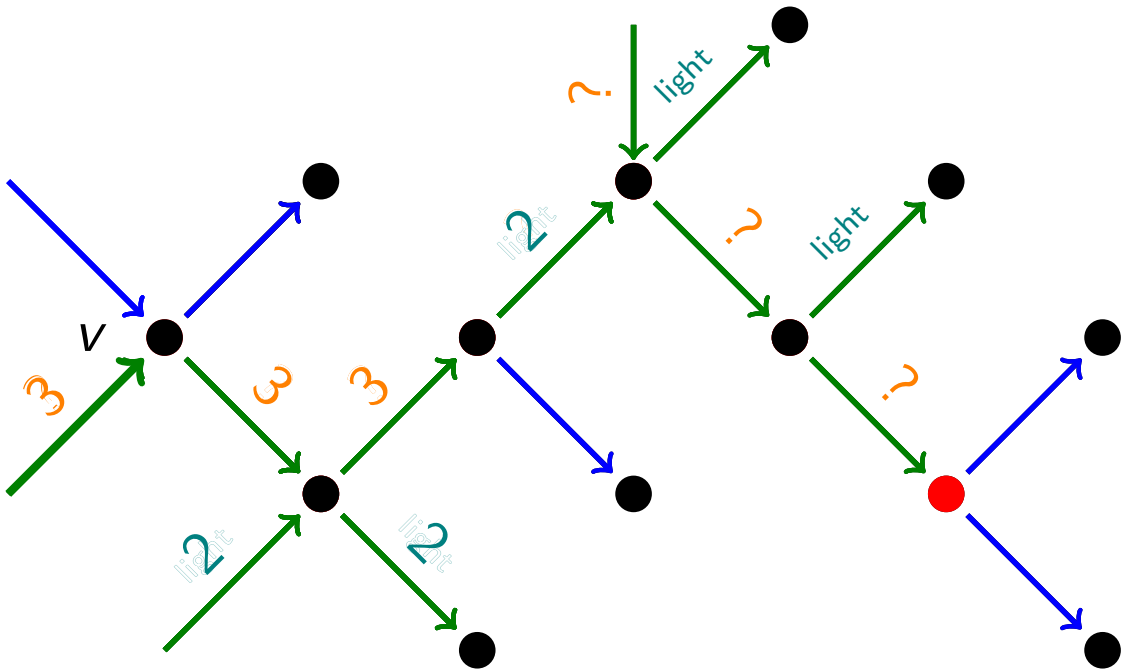

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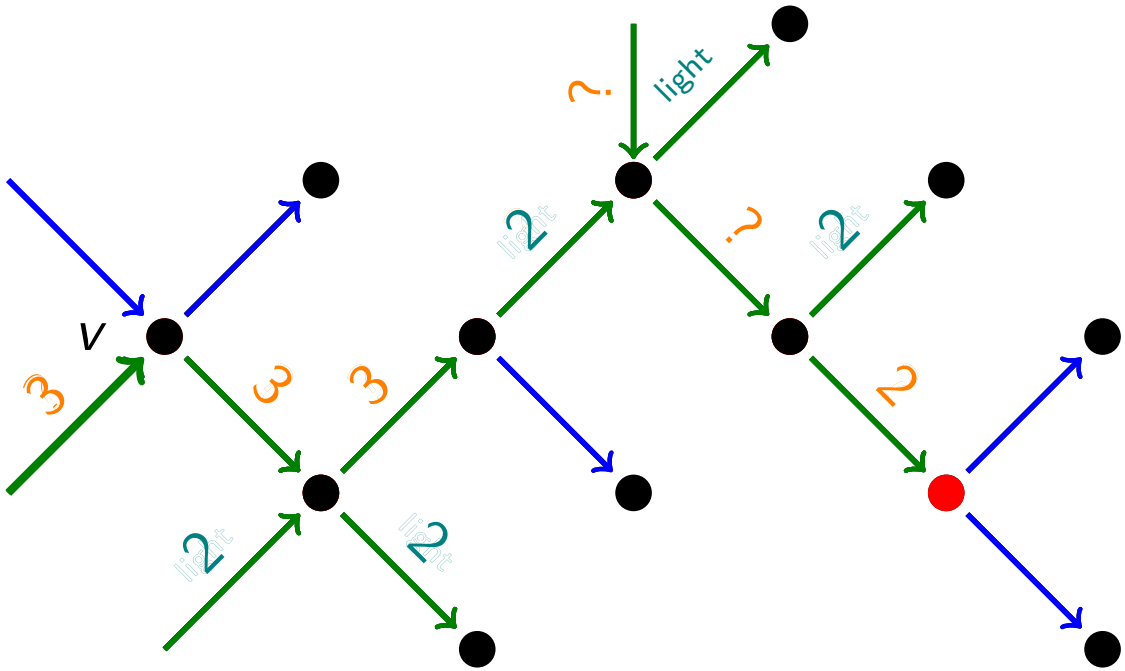

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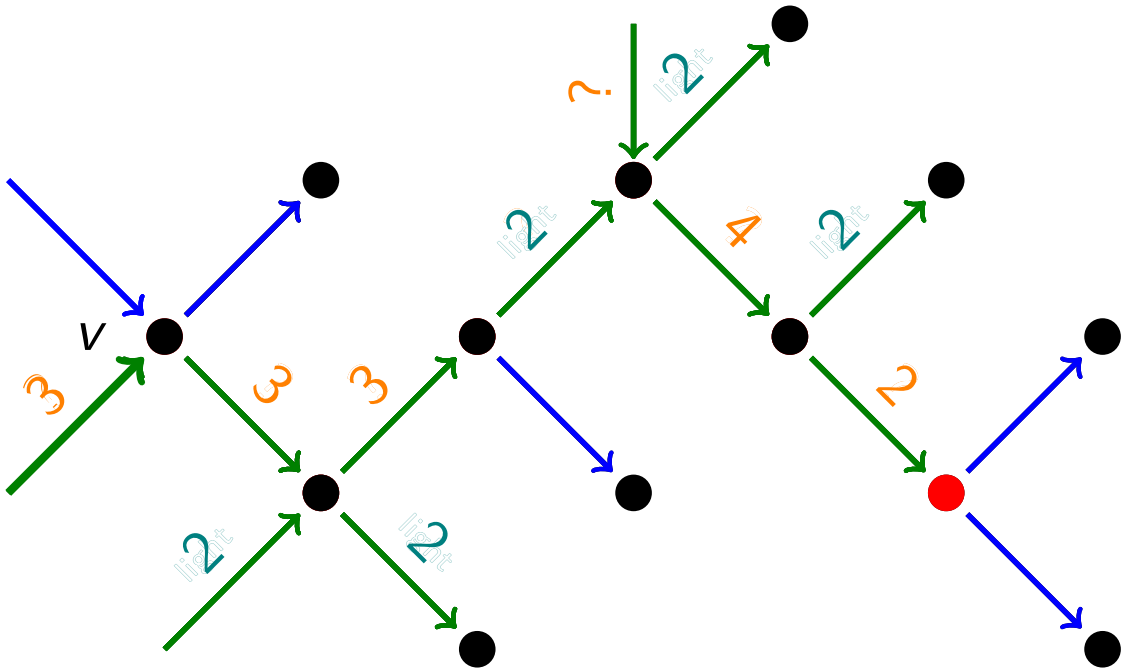

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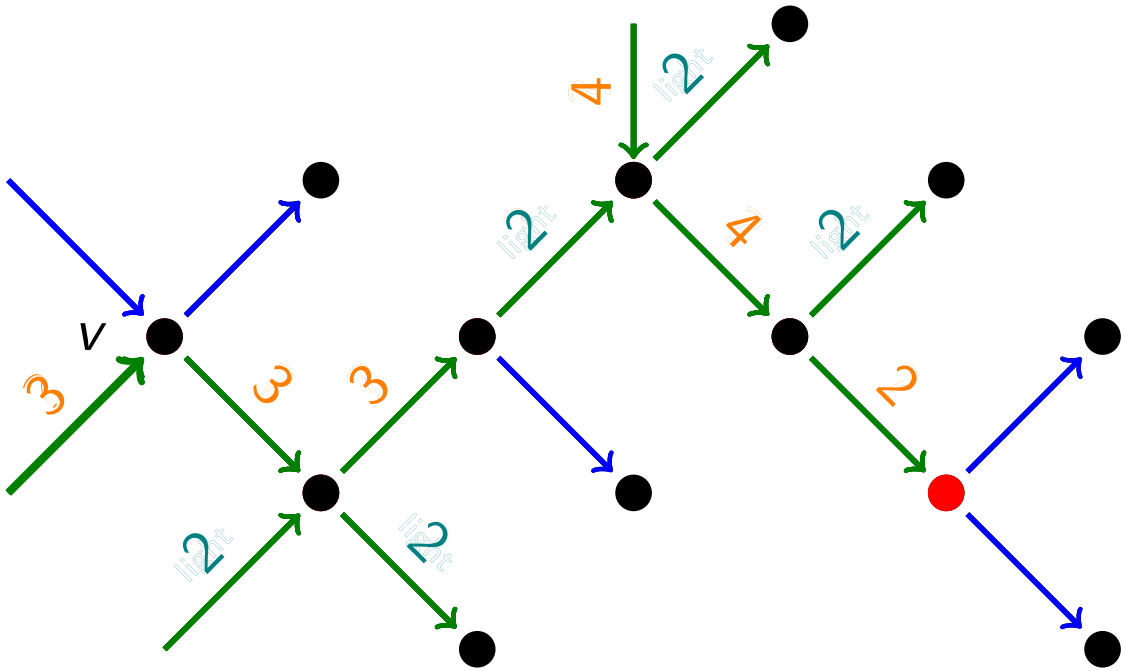
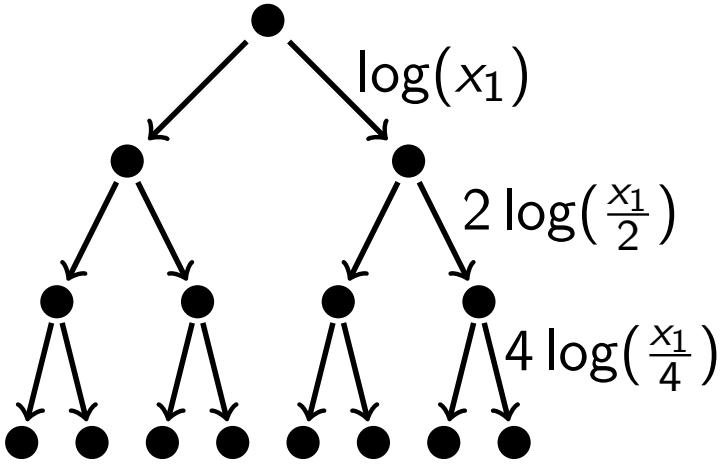
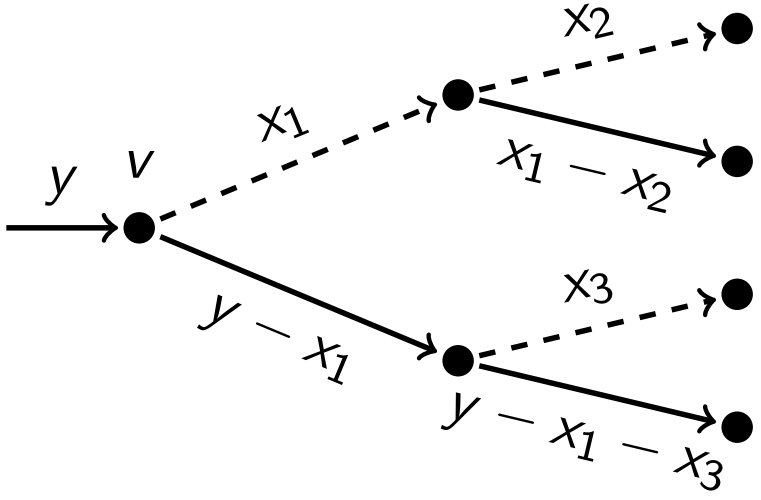


Advice Bits for *light* Edges

For every pair of multi-edges, the algorithm asks for the *light* edge and its number of traversals.





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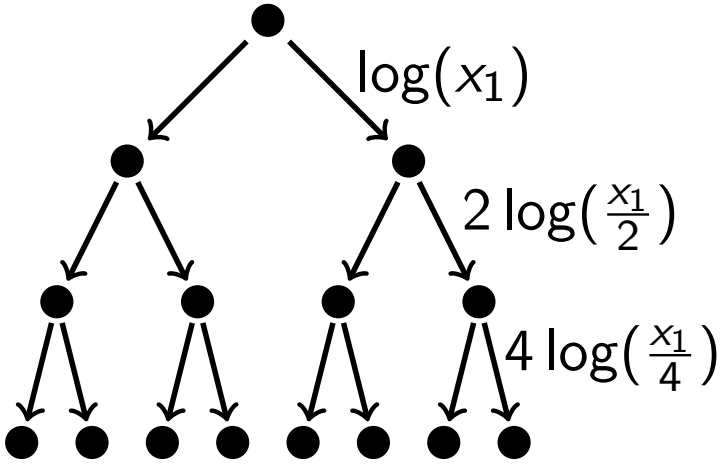

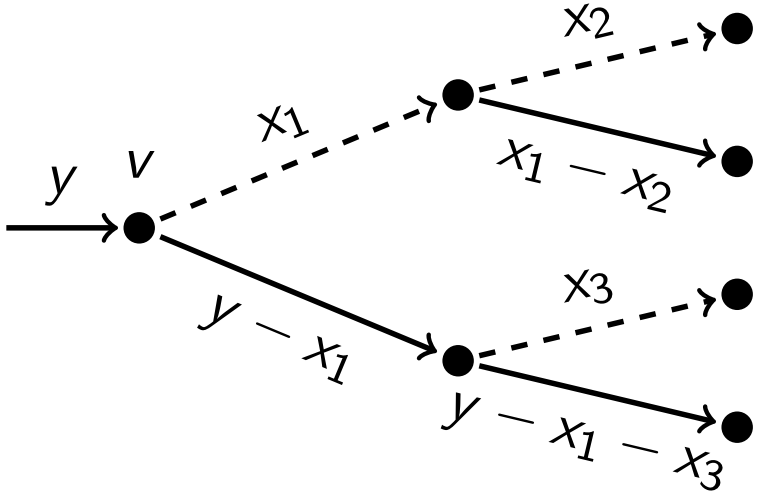

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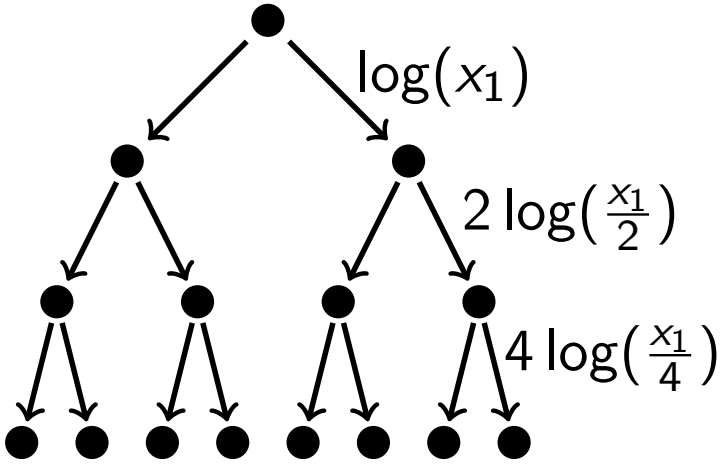

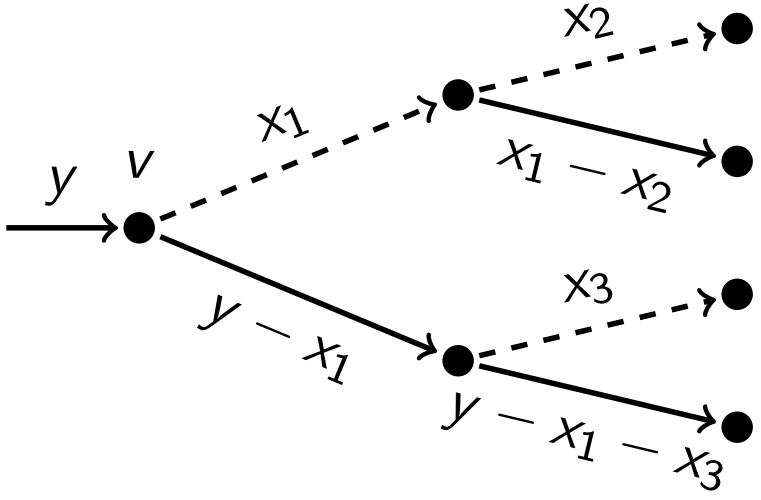

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Per induction: $g(y) \le \frac{5}{2}y$

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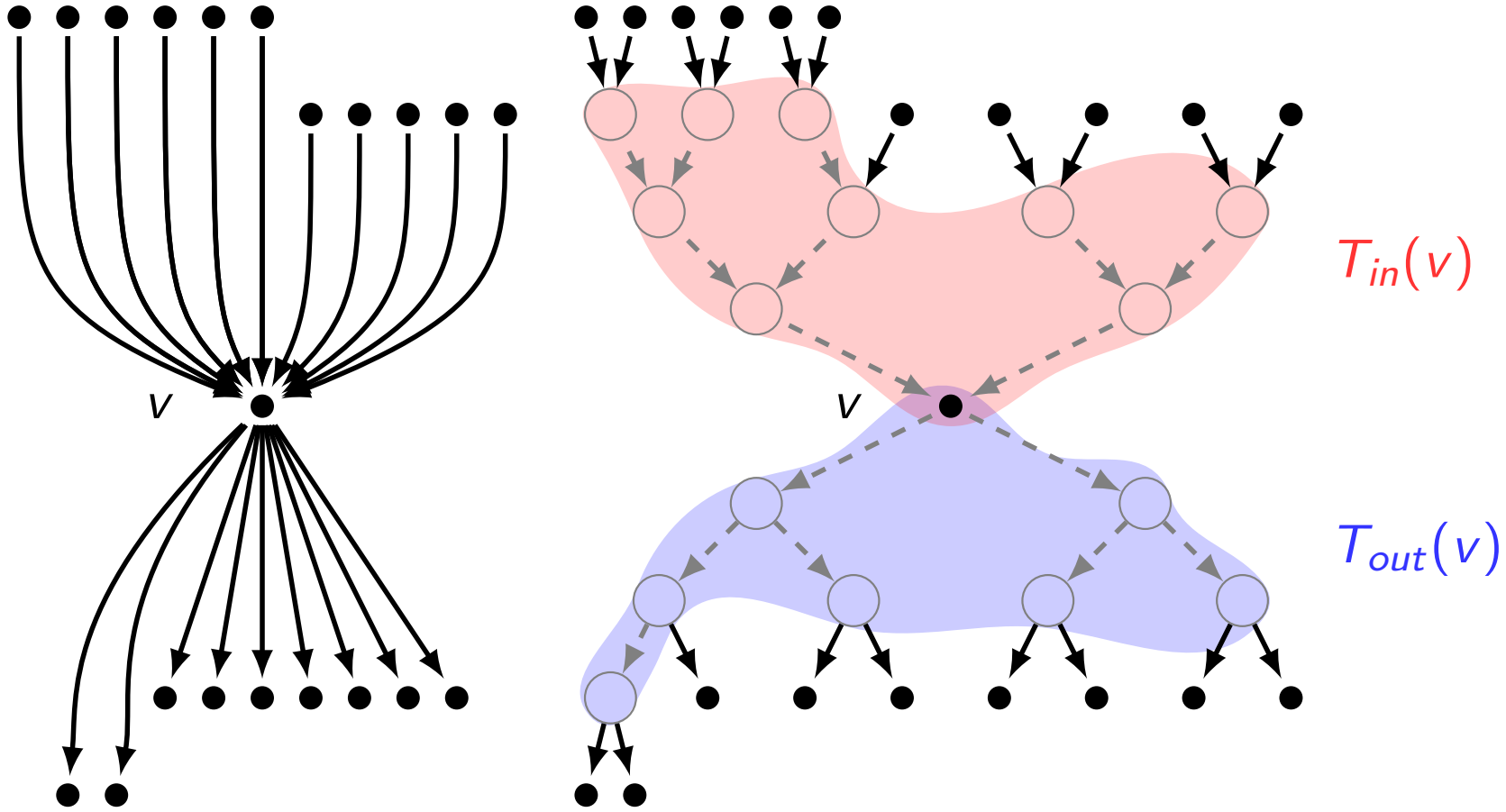
Theorem

There exists an online algorithm which solves the graph exploration problem using $4n + (\log(3) + 5)m$ bits of advice on a given unknown directed graph $G = (V, E)$ with in- and outdegree bounded by 2.

Unbounded Degree Graphs

Oracle and algorithm agree on a transformation of the graph.
Virtual vertices and edges are added to reduce the vertex degree.



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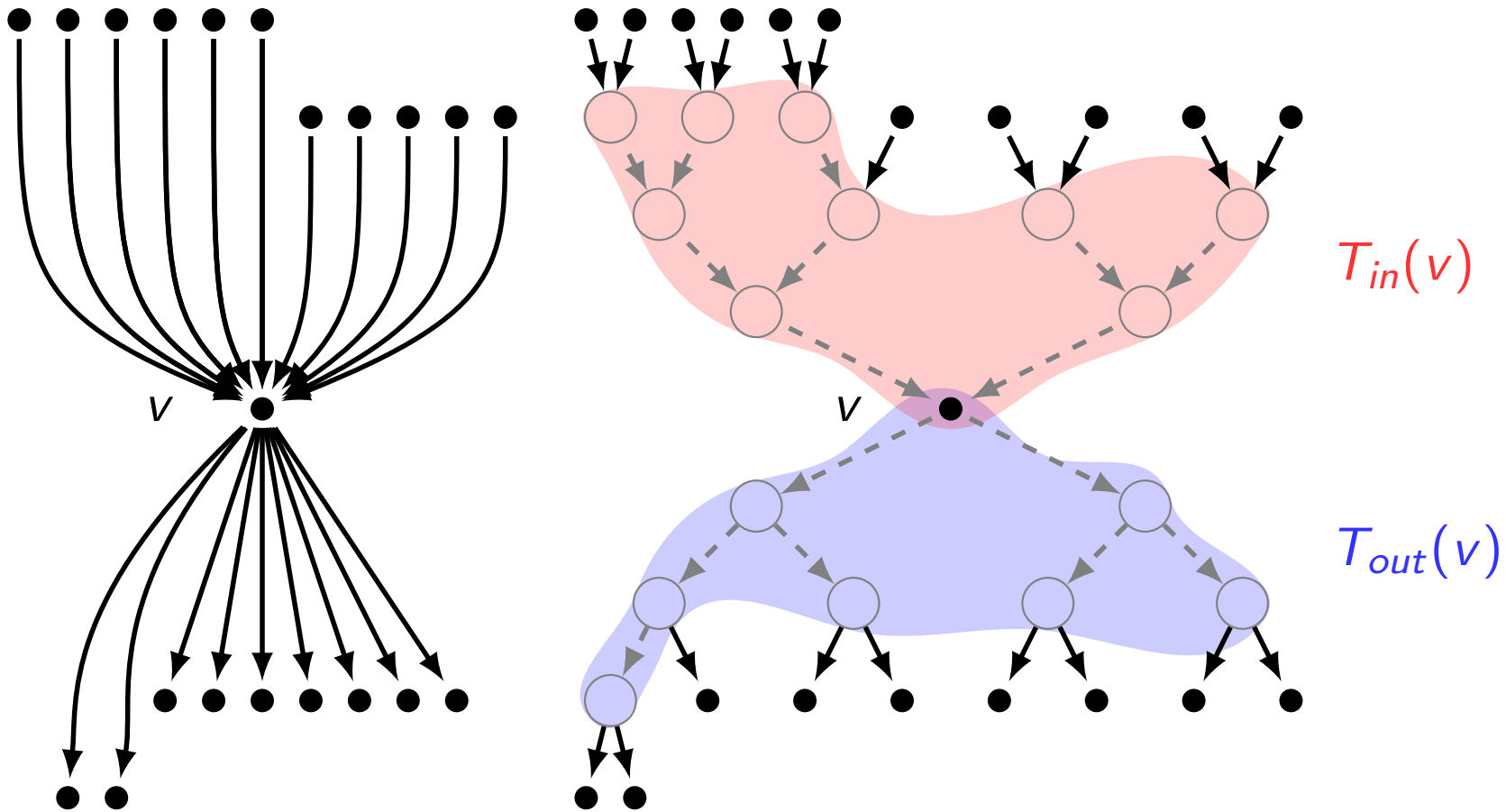
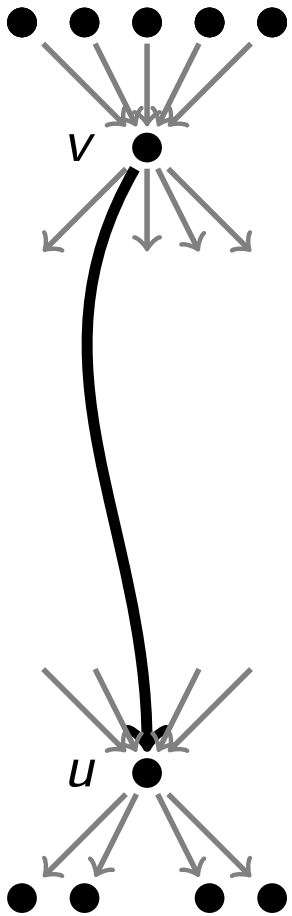
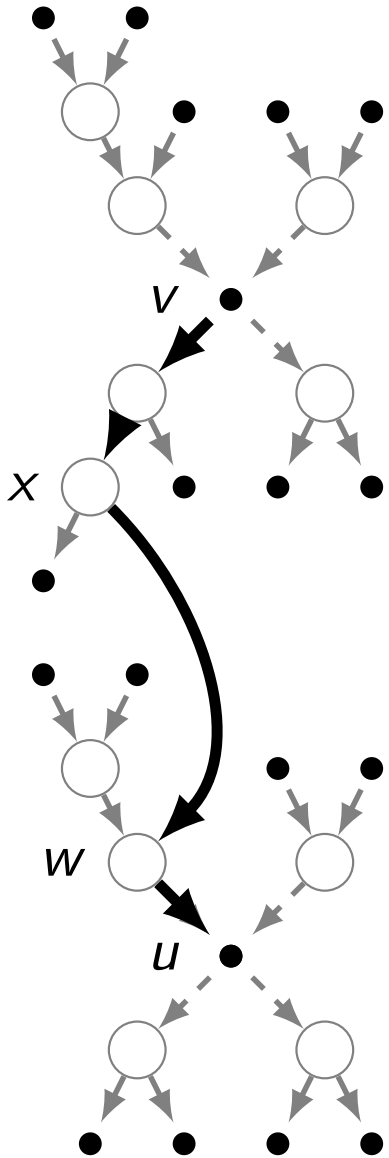


The virtual edges are always in E_{Multi}.

Edges after Transforming the Graph





Lower Bounds with Advice

A technique to prove lower bounds is the partition tree:

Try to find a family of instances $\mathcal{I}$ with the same prefix.
These instances should require different decisions to compute the optimal solution.

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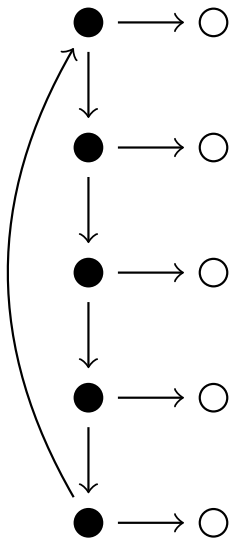
The main argument is that every instance needs a different advice string.
Thus, every algorithm needs at least

$$|\mathcal{I}| = 2^{\log_2(|\mathcal{I}|)} > 2^{b(|\mathcal{I}|)}.$$

Construction

An instance from $\mathcal{I}$ consists of two parts.
Part A consists of cycles with length $x = 5$.



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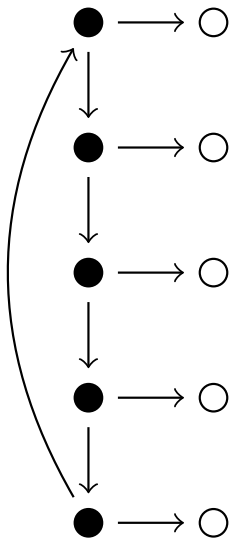

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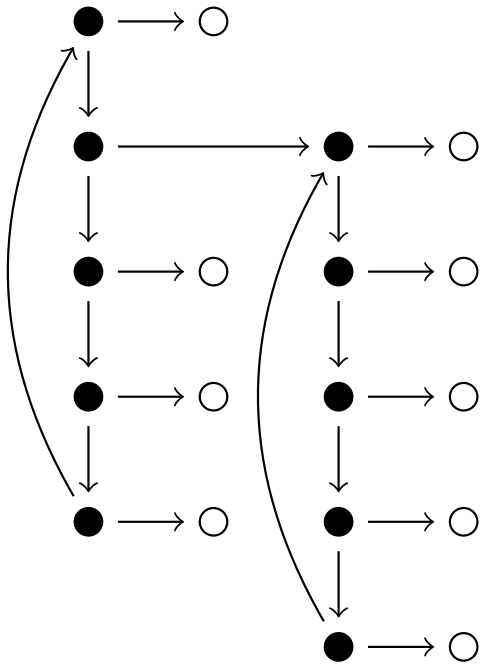

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White vertices are for the second part.

Numbers for the First Part

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It is important that this number is a power of two:

$$|B| = 2^z + 1$$

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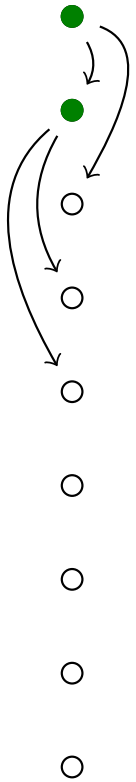

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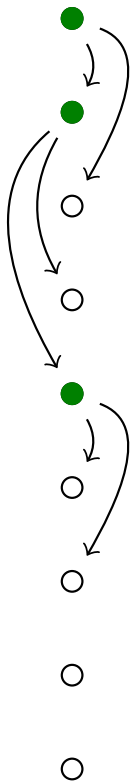

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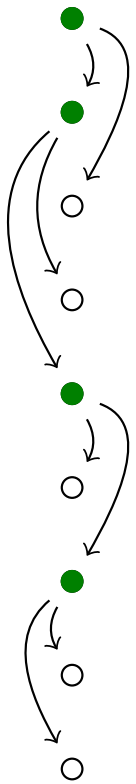

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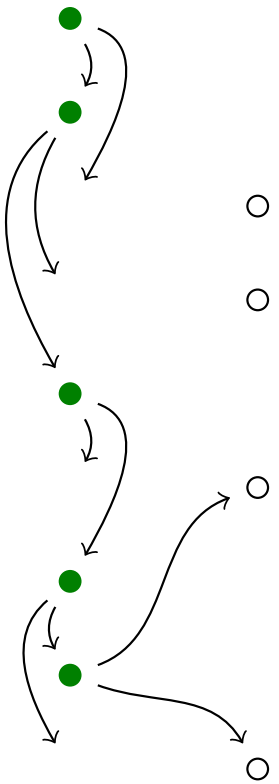

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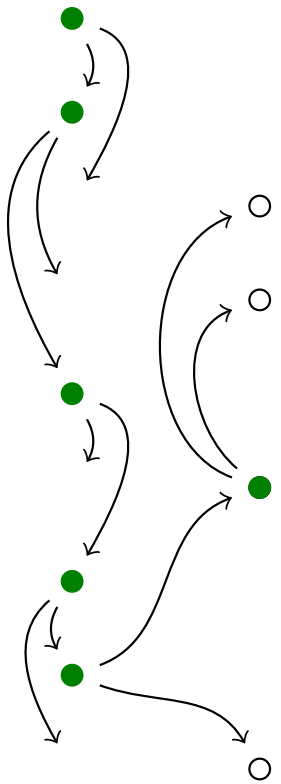

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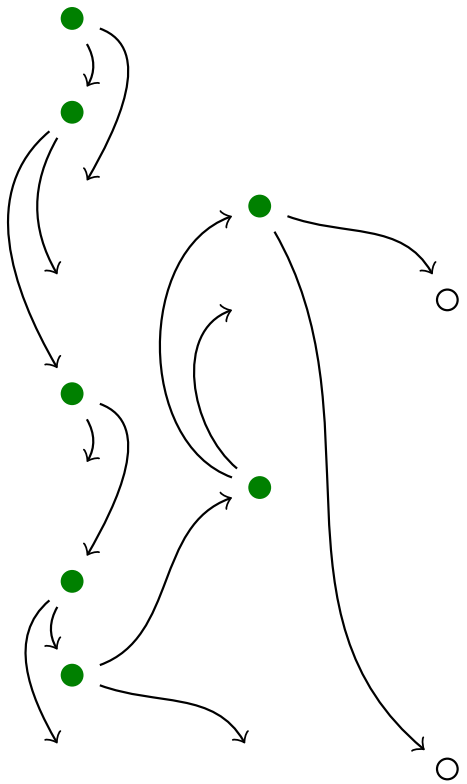

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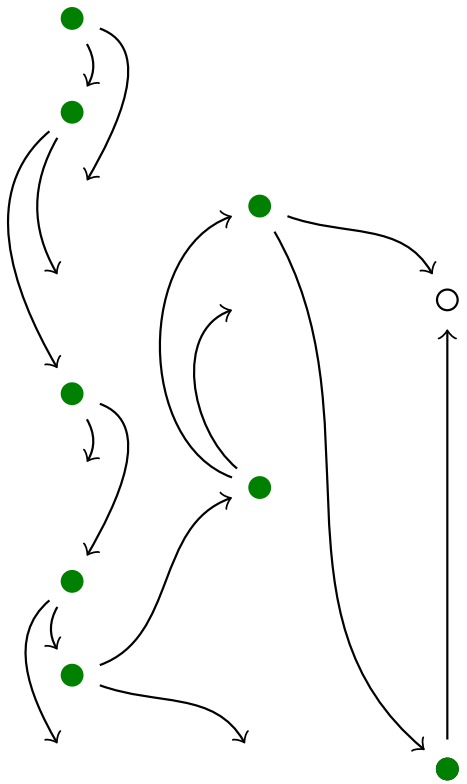

For each vertex we have a binary decision (not the last one)

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Thus, we need slightly more than n bits of advice to solve the lower bound construction optimal.

Results

$n \log(n)$ is the known upper bound for the advice complexity.

Improvement for sparse graphs:

graph model	**exploration**	**upper bound**
directed	cyclic	$2n + 23m$
directed	path	$2n + 23m + \lceil \log(n) \rceil$
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The lower bound is slightly larger than n.

Future Work

The newest result:

Theorem

There exists an online algorithm with advice that uses $5.25n$ bits of advice to solve the graph exploration problem on directed graphs with a bounded outdegree of two.

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Thanks for your attention!